TRANSFORMATION GEOMETRY MODULE

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QUOTES

The Prophet Muhammad (ﷺ – peace be upon him) said: “Acquire knowledge and impart it to the people.” – (Sunan Tirmidhi, Hadith 107)

“Where there is matter, there is geometry”
Johannes Kepler

“Meaning is important in mathematics and geometry is an important source of that meaning”.
David Hilbert

Preface

The writer aims to thank to the Almighty God, Allah, because of His bless and grace, this module titled “Transformation Geometry” can be finished. Moreover, the writer expresses gratitude to the Mathematics Department lecturers, peculiarly, geometry field lecturers.

This module is intended as a completion of geometry literatures which uses English as the medium of instruction. Therefore, it can be used a reference for International Class Program students. Related to the contents, it consists of several transformations on the Euclidean plane, i.e. reflection, halfturn, translation, and rotation. Besides that, there are some concepts related to the transformations concerned in this module namely bijective function, isometry, and the composition of two transformations.

The writer hopes this module is certainly useful for everyone, particulary for Mathematics Department students. However, critiques and advices are emphatically needed for the refinement of this module in future.

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CHAPTER I
TRANSFORMATION

The discussion of transformation geometry commences with the introduction of the concept of function that have been studied in the subject of calculus. The concept underlies the topic of transformation geometry, for example, one of the postulates in Euclidean geometry, i.e. every angle on a plane is associated with exactly one real number.

The function which is discussed here is restricted to the function having domain and origin in the form of V (V is the Euclidean plane). The function definition is given as follows:

Definition: A function of V to V is a mapping that associates every element of V to exactly one element of V

In general, a function is notated with the letter f. If f is a function from V to V that associates every x ∈ V to y ∈ V, it can be written as $y = f(x)$ where x is called the pre-image of y by f and y is called the image of x by f. The origin and the range of the function are V.

In the calculus course, the types of functions are described, however, here only three types are discussed, namely:

a. Surjective Function

A function f is called a surjective function if for every y element V, there exists x element V such that $f(x) = y$. And to show that a function is surjective, we must show that every element (image) has pre-image. In other words, for every element in codomain has pair in domain V.

b. Injective Function

A function f is called injective if for every a and b element of the domain, where $a \neq b$ then $f(a) \neq f(b)$. The statement is equivalent to "if $f(a) = f(b)$ then $a = b$". To show that a function is injective, we must show that for every pair a and b element domain, if $a \neq b$ then we show $f(a) \neq f(b)$, or if $f(a) = f(b)$ then we show $a = b$.

c. Bijective Function

A function f is called bijective if f is surjective function and injective function. So, to show that a function is bijective, we must show that the function is both surjective function as well as injective function.

In Mathematics subject in Junior High School (SMP) and Senior High School (SMA) we have learned about symmetry, rotation, translation, and dilatation. All we have learned are the equivalent of bijective and those are transformations that will be discussed.

Whereas, transformation geometry term can be interpreted as a branch of geometry that discusses transformation, but it can also be interpreted as a geometry which is based on the transformation.

The following discussion presents the geometry in the first interpretation, but at the same time it leads to the second interpretation.

Definition A transformation on plane V is a bijective function of which both the domain and the codomain are V. Where V is the Euclidean plane

To show that a mapping of V to V is a transformation, then the steps which should be consecutively undertaken are checking whether:

1. The mapping is a function.
2. The mapping is surjective.
3. The mapping is injective.

Surjective means that if $\forall$ B $\in$ V, $\exists$ A $\in$ V such that $\mathrm{T(A) = B}$.

B = mapping from A by T, and

A = pre-image of B by T.

$\mathrm{T(A) = A'}$

Injective means that if A and B are elements of the domain, then "$\mathrm{(A \neq B) \Longrightarrow (A' \neq B')}$", where $\mathrm{T(A) = A'}$ and $\mathrm{T(B) = B'}$.

The statement is equivalent to "$\mathrm{(A' = B') \Longrightarrow (A = B)}$"

So, to show that a function is injective, then it must be shown that for each pair of elements of the domain A and B, if $\mathrm{A \neq B}$ then we must show that $\mathrm{A' \neq B'}$, or if $\mathrm{A' = B'}$ then it must be shown that $\mathrm{A = B}$.

A surjective and injective function is a bijective function. Therefore to show whether the function is bijective, it must be shown that whether the function is both surjective and injective.

Example 1:

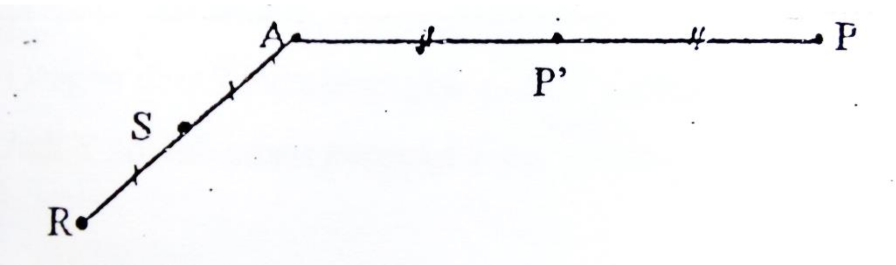
Suppose A ∈ V. There is a mapping T of which both the domain and the codomain are V.

T: V → V is defined as follows:

1) $\mathrm{T(A) = A}$.
2) If $\mathrm{P \neq A}$, then $\mathrm{T(P) = P'}$ with $\mathrm{P'}$ the middle point $\overline{AP}$.

Show that the mapping T is a transformation.

Solution:



The image of point A is the point A itself.

Take a point $\mathrm{R \neq A}$ on V.

Since V is the Euclidean plane, then there is one line passes through the points A and R.

It means that there is only one line segment AR, so that there is exactly one point S with $S \in \overline{AR}$, such that $\mathrm{AS} = S\mathrm{R}$.

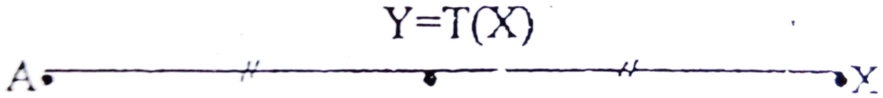
Since R is an arbitrary point, it means that for each X ∈ V, there is Y ∈ V, where $\mathrm{Y = T(X)}$.

Therefore T is a function

Is T surjective?

In other words, does every point in V has a pre-image?

To answer these questions, it must be shown that for arbitrary point Y ∈ V, is there X ∈ V so that $\mathrm{T(X) = Y}$.



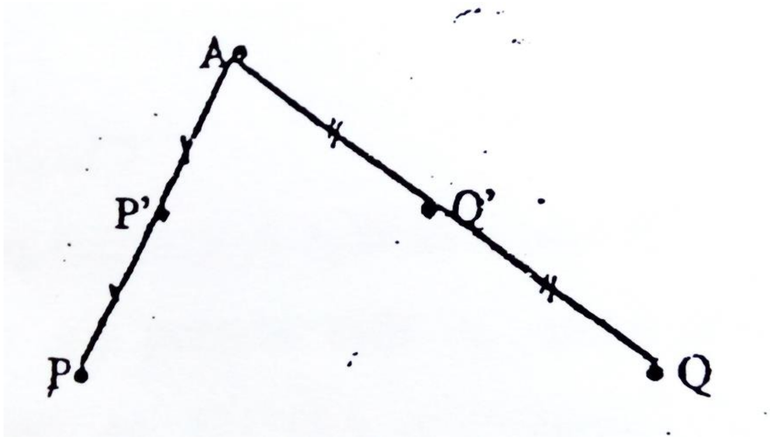
According to the first condition, if $Y = A$, its pre-image is A itself, because $T(A) = A$. If $Y \neq A$, since V is the Euclideanplane, then there is exactly one X with $X \in \overline{AY}$ such that $AY = YX$. It means that X is pre-image of the point Y.

Thus, every point in V has a pre-image which implies that T is a surjective mapping.

Is T injective?

To show that T is injective, take any point $P, Q \in V$ with $P \neq Q$.

- The first case if P, Q and A are not collinear.
  It will be shown that the position of $P' = T(P)$ and $Q' = T(Q)$



Suppose $P' = Q'$.

Since $P' \in \overline{AP}$ and $Q' \in \overline{AQ}$, then $\overline{AP}$ and $\overline{AQ}$ has two intersection points, namely point A and point P' or Q'. It means $\overline{AP}$ and $\overline{AQ}$ coincide Since $Q \in \overline{AQ}$ and $\overline{AP} = \overline{AQ}$ then $Q \in \overline{AP}$, resulting in point P, Q and A are collinear.

It is contrary to the fact that P, Q and A are not collinear. It means that the assumption that $P' = Q'$ is not true, and it should be $P' \neq Q'$.

So if $P \neq Q$ then $P' \neq Q'$, which means injective.

- The second case is if P, Q and A are collinear (it is also injective)

From the proof above, it turns out that T is an injective mapping.

Because T is a surjective and injective mapping then T is bijective. So T is a transformation.

Example 2:

Given relation $T\ [(x, y)] = (2x + 1, y - x)$.

Show that this mapping is a transformation.

Solution:

If $P(x, y)$ then $P' = T(P) = (2x + 1, y - x)$. It means that the domain of T is the whole plane V.

Is T surjective?

Take an arbitrary point $A(x, y)$, is A a domain of T ?

Suppose that $B(x', y')$ is a domain of point A, then certainly $T(B) = A$ or $T[(x', y')] = (x, y) \leftrightarrow (2x' + 1, y' - x') = (x, y)$

we get $x' = \frac{x-1}{2}$ and $y' = \frac{2y+x-1}{2}$.

So $B = (\frac{x-1}{2}, \frac{2y\pm1}{2})$, hence $T(\left(\frac{x-1}{2}, \frac{2y\pm1}{2}\right) = (x, y)$.

Since (x', y') always exists for each (x, y) then B (domain of A) always exists meaning that $T(B) = A$. Because A is an arbitrary point in V, then each point in V has domain which means that T is surjective.

Is T injective?

Take points $P(x_1, y_1)$ and $Q(x_2, y_2)$ where $P \neq Q$

Is $P' \neq Q'$?

Suppose that $P' = Q'$ then $(2x_1 + 1, y_1 - x_1) = (2x_2 + 1, y_2 - x_2)$

Since $x_1 = x_2$ and $y_1 = y_2$ then $P = Q$. It is contrary to the fact that $P \neq Q$.

It means that the assumption that $P' = Q'$ is false and it should be $P' \neq Q'$.

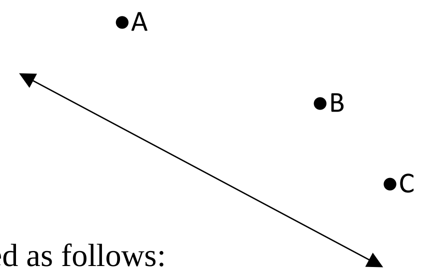
It is now proven that if $P \neq Q$ then $P' \neq Q'$. So, T is injective.

Therefore T is injective and surjective, in other words, T is a transformation

Excercises

1) Let point K and line segment of $\overline{AB}$ where K $\notin \overline{AB}$ and a line g so that g // $\overline{AB}$ and the distance between K and $\overline{AB}$ is two times the distance between K and g. There is a mapping T with domain $\overline{AB}$ and the range g so that if P $\in \overline{AB}$ then T(P) = P’ = $\overline{KP}$ ∩ g.
   a. What is the range of P’ if P moves through $\overline{AB}$.
   b. Prove that T is injective.
   c. If E and F are two points on $\overline{AB}$, what can be interpreted about the distance of E’F’, if E’ = T(E) and F’ = T(F).
2) Let $\mathrm{O}(0,0)$, $\mathrm{C}_1 = \left\{(\mathrm{x},\mathrm{y}) \,\middle|\, \mathrm{x}^2 + \mathrm{y}^2 = 1\right\}$, $\mathrm{C}_2 = \left\{(\mathrm{x},\mathrm{y}) \,\middle|\, \mathrm{x}^2 + \mathrm{y}^2 = 25\right\}$.

   T: $\mathrm{C}_1 \rightarrow \mathrm{C}_2$ is a mapping defined as: “if P $\in$ C_1, then T(P) = P’ $\overrightarrow{\mathrm{OP}} \cap \mathrm{C}_2$
   a. If A(0,1), determine T(A).
   b. Find the domain of B(4,3).
   c. If D is any point on domain T, find the distance between DD’, D’ = T(D).
   d. If E and F are two points on the domain of T. What can be interpreted about the distance between E′F′?
3) $Let\ F : V -> V, if\ P\ (x, y)\ then\ f(P) = (|x|, |y|)$
   a. Determine $f(A)$ if $A(-,6)$
   b. Determine all the pre-images of the point $B(4,2)$
   c. What is the shape of the range?
4) Let function f : axis $x -> v$ defined as: “if $P(x,0)$ then $f(P) = (x, x^2)$
   a. Find the image of $A(5,0)$ by f
   b. Is $B(-13,169)$ $\in$ image of f
   c. Is f surjective?
5) Let a line s and point of A, B, C as shown below



   $T: V \rightarrow V$ is defined as follows:
   i) if $P \in S$ then $(P) = P$
   ii) if $P \notin S$ then $T(P) = p’$ such that is axis of $\overline{PP'}$

a. Draw $A' = T(A), B' = T(B)$

b. Draw the pre-image of point C

c. is T a tranformation?

d. Prove that $A'B' = AB$

6) Let two lines g and h are parallel in the Euclidean plane V, and a point of A which is in the middle between g and h. T is the image of the domain of g defined as : $if\ P \in g\ then\ P' = T(P) = \overline{PA} \cap h$

   a. What is the range of T

   b. If $B \in g$, $C \in g$, and $B \neq C$, $B'C = BC\ with\ B' = T(B), C' = T(C)$

   c. is T injective?

7) Let three different points A, E, D not collinear and a relation T defined as:
   $T(A) = A, T(P) = P'$, such that P is the middle point of $\overline{AP}$.

   a. Draw $E' = T(E)$

   b. Draw Q so that $T(Q) = D$

   c. is T a transformation?

CHAPTER II

REFLECTION

In senior high school physics, we have learned about the properties of reflection. It says that if an object is x units in the front of a mirror, then the image of the object is also located as far as x units behind the mirror, and if the object is located on the mirror, its image will coincide with the object. It is further discussed geometrically in this reflection section:

Definition:
Reflection in line s is a function M_s that is defined for each point P on plane V as follows:

i. If $P \in s$ so $M_s = P$.
ii. If $P \notin s$ so $M_s(P) = P'$, in a way such that s is the axis of $\overline{PP'}$.

Reflection in line s is notated M_s. Line s is called the axis of reflection or mirror line.

As shown in the previous section, to show whether reflection is a transformation, it must be shown whether the reflection is a function that is surjective and injective.

To show that a reflection is a transformation, the steps which should be undertaken are answering the following questions:

1. Is reflection a function?
   Based on its definition, reflection is a function from V to V.
2. Is reflection surjective?
   Take arbitrary point A 'on V. If $A' \notin s$. Geometrically, A is the element of V so s become the axis $\overline{AA'}$ (since V is an Euclidean plane).
   It means that the $M_S(A) = A'$ implying that every A' has pre-image. Then, M is surjective
3. Is reflection injective?
   Take two arbitrary points $A, B \in V$ where $A \neq B$.

There are three possibilities, namely:

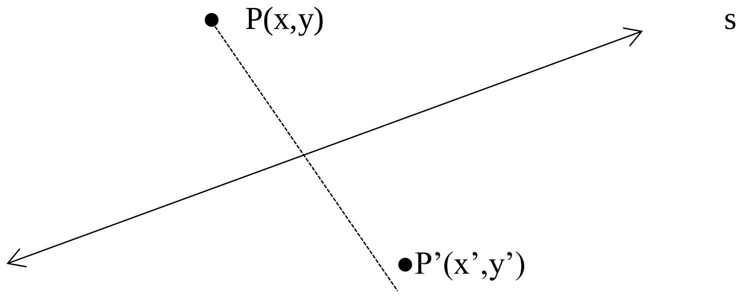
a. $A \in s$ dan $B \in s$
   It means $M_s(A) = A' = A$ and $M_s(B) = B' = B$
   Since $A \neq B$ it means $A' \neq B'$
   Then M is injective.
b. $A \in s$ and $B \notin s$
   It means $Ms(A) = A' = A$ and $M_s(B) = B'$ such that s is the axis $\overline{BB'}$.
   Because $A \in s$ and $B' \notin s$, so that $A' \neq B'$
   Then M is injective.
c. $A \notin s$ and $B \notin s$
   Assume that $M_s(A) = M_s(B)$ or $A' = B'$. Since $\overline{AA'} \perp s$ and $\overline{BB'} \perp s$, so $\overline{BA'} \perp s$, thus it is obtained that from point A', two distinct lines can be created perpendicular to line s which is impossible. Then the assumption that $M_S(A) = M_s(B)$ or A '= B' is false. Thus, it should be $A' \neq B'$.
   Thus, if $A \neq B$ then $A' \neq B'$.

Then, M is injective.

Since M_S is a function that is both surjective and injective, M_s is transformation.

Theorem: Every reflection in a line is a transformation

Suppose that s: ax + by + c = 0 , and P (x,y), where M_s(P) = P'(x', y').



If P $\notin$ s so $\overline{PP'} \perp$ s , then:

$\frac{y'-y}{x'-x} = \frac{b}{a}$ .... (i)

If Q is the midpoint of $\overline{PP'}$, then Q($\frac{x+x'}{2}$, $\frac{y+y'}{2}$) lies on the line s.

So a($\frac{x+x'}{2}$) + b($\frac{y+y'}{2}$) + c = 0................(ii)

From the equation (i) and (ii), it is obtained that:

$$x' = x - \frac{2a(ax+by+c)}{a^2+b^2}$$

$$y' = y - \frac{2a(ax+by+c)}{a^2+b^2}$$

so, if s: ax + by + c = 0, and P(x,y), then M_s(P) = P'(x',y') where

$$x' = x - \frac{2a(ax+by+c)}{a^2+b^2}$$

$$y' = y - \frac{2a(ax+by+c)}{a^2+b^2}$$

Example 1:

If on V there is an orthogonal axis system with A (1,3) and B (-2,1). Determine the equation of the line s so M_s (A) = B!.

Solution:

M_s(A) = B means that s is the axis $\overline{AB}$. If T is the midpoint of $\overline{AB}$, then s passes through the point T and perpendicular to $\overline{AB}$. m$\overleftrightarrow{AB}$ = $\frac{2}{3}$ => $_m$S = - $\frac{3}{2}$, and T (-$\frac{1}{2}$, 2).

So, the line s : y – 2 = - $\frac{3}{2}$ (x + $\frac{1}{2}$) or

s : 3x + 2y - 2$\frac{1}{2}$ = 0 or s : 6x + 4y – 5 = 0

Example 2 :

Suppose line s : 2x – 3y + 5 = 0

a. Determine M_s(A) if A(2,-5)
b. Determine M_s(O)

Solution:

Given line s : 2x – 3y + 5 = 0 => a=2 , b=-3 , and c=5

a. A(2,-5) , M_s(A) = A'(x',y')

$$x' = x - \frac{2a(ax+by+c)}{a^2+b^2}$$

$$x' = 2 - \frac{2.2[2.2+(-3).(-5)+5]}{2^2+(-3)^2}$$

$$= 2 - \frac{96}{13}$$

$$= \frac{-70}{13}$$

$$y' = y - \frac{2b(ax+by+c)}{a^2+b^2}$$

$$y' = -5 - \frac{2.(-3)[2.2+(-3).(-5)+5]}{2^2+(-3)^2}$$

$$= -5 - \frac{-144}{13}$$

$$= \frac{79}{13}$$

Thus, M_s(A) = A' ($\frac{-70}{13}$, $\frac{79}{13}$)

c. M_s(O) = O'(x',y')

$$x' = x - \frac{2a(ax+by+c)}{a^2+b^2}$$

$= 0 - \frac{2.2[2.0+(-3).0+5]}{2^2 + (-3)^2}$

$= 0 - \frac{20}{13}$

$= \frac{-20}{13}$

$y' = y - \frac{2b(ax+by+c)}{a^2 + b^2}$

$= 0 - \frac{2.(-3)[2.0+(-3).0+5]}{2^2 + (-3)^2}$

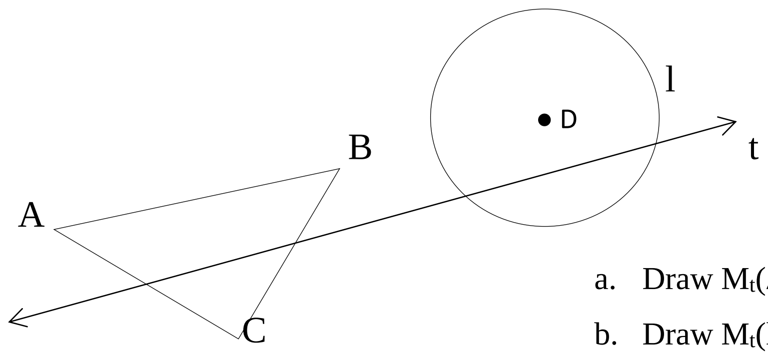
$= 0 - \frac{(-30)}{13}$

$= \frac{30}{13}$

Thus, $M_s(O) = O'\left(\frac{-20}{13}, \frac{30}{13}\right)$

Exercises :

1. Suppose line g = {(x,y)|y = x}
   a. If A(2,-3), determine $M_g(A)$.
   b. If B'(-3,5), determine pre-image of B' by M_g.
   c. If P(x,y) any point, determine $M_g(P)$.
2. Suppose h = {(x,y)|y = 2}
   a. If C(3,$\sqrt{2}$), determine C'.
   b. If D'(2,-4), determine pre-image D, by M_h.
   c. If P(x,y), determine P'.
3. Suppose s = {(x,y)|x = -3}
   a. If A(4,1), determine A' = $M_s(A)$.
   b. Determine the coordinate of point C if $M_s(C)$ = (-2,7).
   c. If P(x,y) is any point, determine $M_s(P)$.
4. Suppose line l = {(x,y)|2x + 3y = 11}
   a. Determine $M_l(O)$.
   b. Determine $M_l(E)$ with E(1,2).
   c. If F(x, 2x-1), determine the coordinate F if $M_l(F)$ = F.
5. Suppose line s = {(x,y)|2x + y = 1} and t = {(x,y)|x = -2}. Find the equation of line s' = $M_t(s)$.
6. Suppose line t, circle l with center D, and Δ ABC as shown below :



a. Draw $M_t(\Delta\ ABC)$

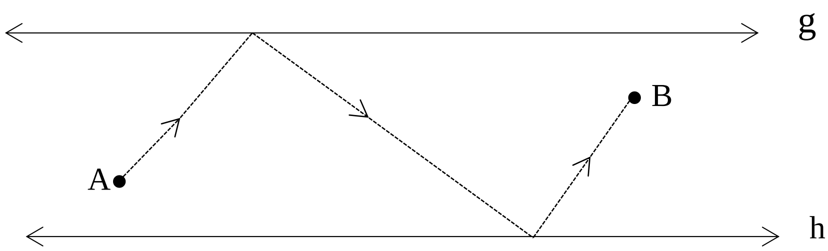
b. Draw $M_t(l)$

7. If lines g = {(x,y)|y = 1}, h ={(x,y)|y = x} and k ={(x,y)|x = 3}. Find the equation of the following lines :

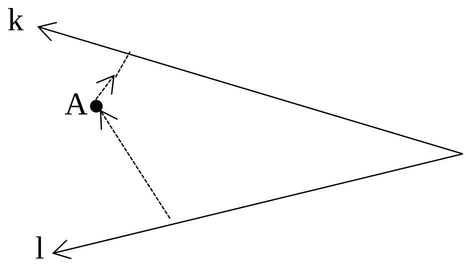
    a. $M_g(h)$ c. $M_h(g)$

    b. $M_g(k)$ d. $M_h(k)$

8. M_k is a reflection that connects point A(4,8) to point B(8,0). Determine the equation (image) of circle $(x + 1)^2 + (y – 3)^2 = 9$, if it is reflected to the line k.
9. If two points P and Q. Draw a line t so that $M_t(P) = Q$ and determine $M_t(Q)$.
10. There are an orthogonal axis system in V, point A(1,3), and point B(-2,-1). Determine the equation of a line of g so that $M_g(A) = B$.
11. Given two parallel lines g and h, points A and B as shown in the figure. Draw the shortest path from A to B providing that it must be reflected on g then on h.



12. If g = {(x,y)|y = -x} and h ={(x,y)|3y = x + 3}
    Show that whether point A(-2,-4) lies on line h' = M_g (h).
13. If line g ={(x,y)|6x – 3y + 1=0} and a point A(k,2).
    Find the value of k if $M_g(A) = A$.
14. Two walls form an angle as shown in the figure of which it is formed by line k and line l. A ball is located in the point A. Sketch where the ball should be directed such that if it is reflected on k and on l, it will be bounced back to A.



15. Given line $g = \{(x,y)|3x - y + 4 = 0\}$ and $h = \{(x,y)|2x + 3y = 6\}$.
    Determine the equation of line $g' = M_h(g)$, and $h' = M_g(h)$.

CHAPTER III

ISOMETRY

In daily life, many events or movements are transformations such as the movement of a table and the opening or closing of a door. The movement of a table from one place to another and the opening or closing of door do not alter the length and the width of table or doors except the position of the table or the door. Such kind of transformation is called isometry.

Definition
A transformation T is an isometry if for every pair of points P, Q satisfies $P'Q' = PQ$, where $P' = T(P)$ and $Q' = T(Q)$.

If $A' = M_s(A)$, $B' = M_s(B)$, then $A'B' = AB$. (prove!)

Theorem

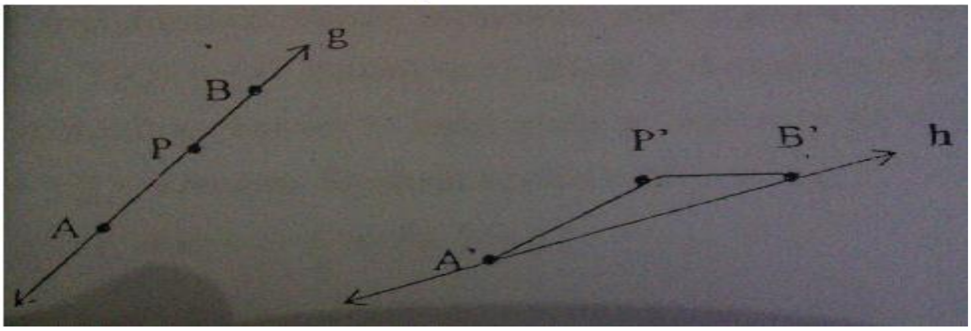
Any reflection on a line is an isometry.

As being previously discussed, the result of a reflection is preserving the lenght of segment or the distance between two points, thus reflection is isometry.

Besides preserving the distance between two points, isometry also has the properties as follows:

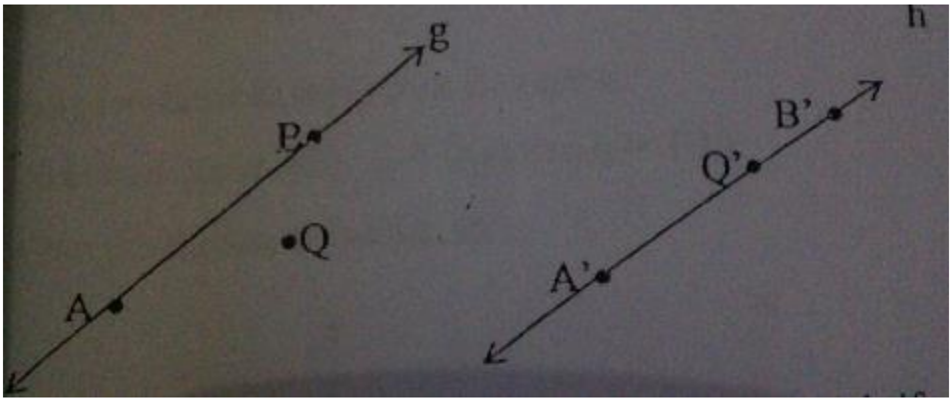
a. Mapping a line into a line

Suppose g is a line and T is an isometry, it will be proved that $g' = T(g)$ is a line.
Put the points A and B on the line g ($A \in g$ and $B \in g$).
Suppose $T(A) = A'$ and $T(B) = B'$, create a line h through point $A'$and B'. It will be proved that the line $h = g'$



Take an arbitrary point P on g such that it forms APB, and let $P' = T(P)$. On the line g, $AP + PB = AB$.
Because T is an isometry then $A'B' = AB$, $A'P' = AP$, $P'B' = PB$.
Suppose P' lies on outside of h, then at $\Delta A'P'B'$ must be satisfied that $A'P' + P'B' > A'B'$ since $A'B' = AB$, $A'P' = AP$, $AP + PB > AB$

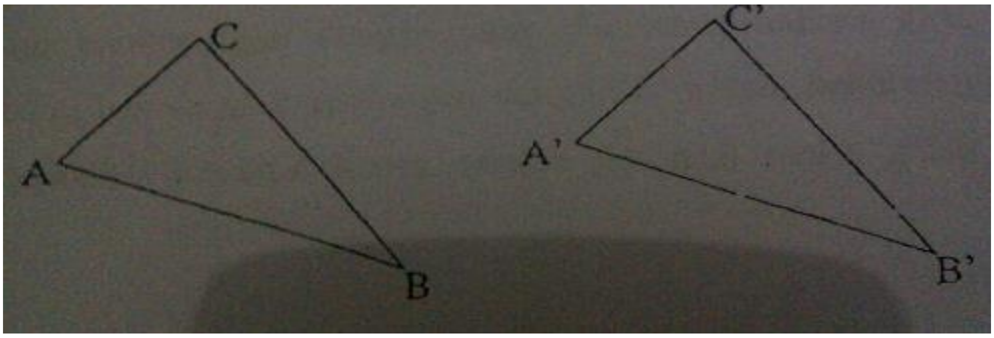
It contradicts to the assumption that $AP + PB = AB$.
It means that the assumption that P' lies on outside of the h is not true. Suppose P' lies on h or $A'P'B'$.
So $g' \subset h$
The reverse direction is proved in the same way by assuming that Q' is any point on h.



Since T is a transformation, it means that it satisfies the surjective properties, then there is a Q such that $T(Q) = Q'$. Suppose Q lies outside of g. Using the triangle inequality, it can be proved that Q should be on g, so that $Q' = T(Q)$ must be on $g' = T\ (g)$. Thus, it means that $h' \subset g'$.
Since $g' \subset h$ and $h \subset\ g'$ then $h = g'$

b. Preserving the size of the angle between two lines

Take the three points A, B, and C which are not collinear.
$A' =\ T(A)$, $B' =\ T(B)$ and $C' =\ T\ (C)$

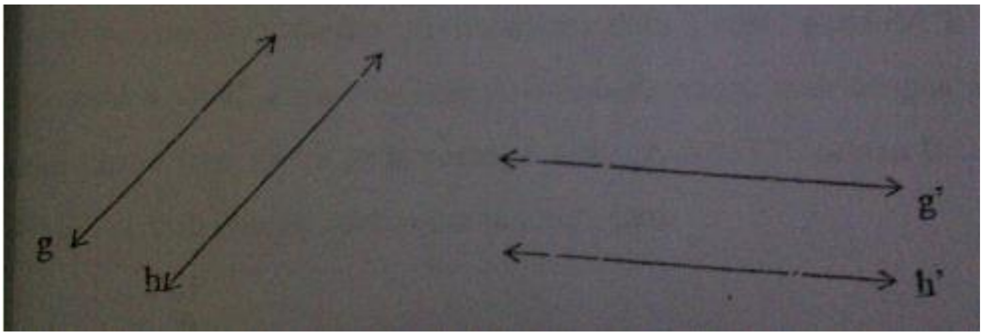


See the ΔABC. According to (a), since the AB and BC are straight lines so $A'B'$ and $B'C'$ are also straight lines.
Because T is an isometry then $A'B' = AB$, $B'C' = BC$, and $A'C' =\ AC$.
It means that $\Delta ABC\ \cong\ \Delta A'B'C\ '$(s.s.s), which also means that the vertices of the triangles are in the same position and the same magnitude.
Thus, isometry preserves the angles.

c. Preserving the parallels of two lines

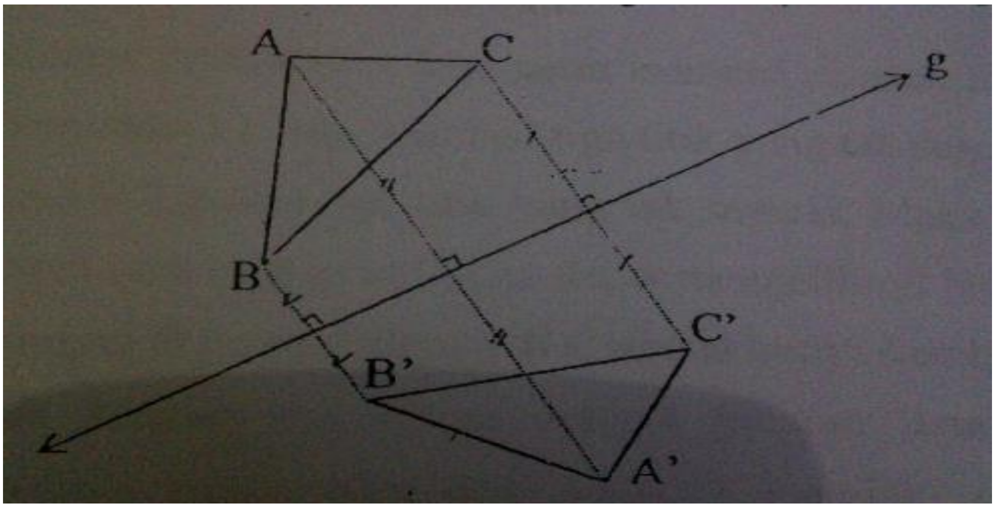
Given Line $g \ // \ h$, $g\ ' = \ T\ (g)$ and $h' \ = \ T\ (h)$.

It will be proved that $g\ '//\ h'$

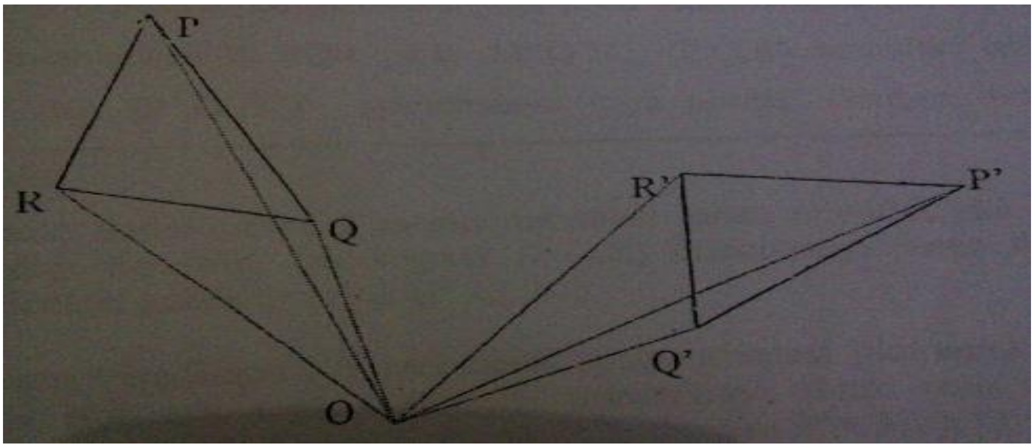


Suppose the line g 'intersect h' at the point P', then $P' \in g'$ and $P' \in h'$. Since T is a transformation then for $P' \in h'$ there must exist P so that $T(P) = P'$ where $P \in g$ and $P \in h$. It then implies g and h intersect at point P.

It contradicts to the assumption that $g \ // \ h$ implying the assumption that g 'intersects h' is wrong, it should be $g'//\ h'$. One consequence of that property is that if $g \perp h$ then $g' \perp h\ '$, where T is an isometry, $g' \ = \ T(g)$, and $h' = \ T(h)$. If ΔABC are reflected to the line g, the map is $\Delta A'B'C'$, or $M_g(\Delta ABC) = \Delta A'B'C'$



A reflection in the line g maps the ΔABC on $\Delta A'B'C'$. If the ΔABC with the order of the $A - B - C$ is opposite to the clockwise direction, then the image which is $\Delta A'B'C'$ having the order $A' - B' - C'$ is clockwise direction.



Meanwhile, the rotation about the center of rotation O maps ΔPQR on $\Delta P'Q'R'$. If on the ΔPQR, the direction of $P-Q-R$ is clockwise direction then its image $\Delta P'Q'R'$, has also clockwise direction $P'-Q'-R'$.

Concerning the further discussion of isometry phenomenon above, we shall introduce the concept of orientation of the pair of three points which are not collinear.

Suppose (A,B,C) is the pair three points which are not collinear. Then through A, B, and C, there is exactly one circle I. We can circumnavigate I started from A, then in B, then in C and ended up back at A.

If the circumferential direction is the same as the clockwise direction, it is said that the pair of three points (A,B,C) has opposite orientations to clockwise direction or negative orientation.

It means that the reflection of the line g that maps ΔABC into the $\Delta A'B'C'$, (A,B,C) has negative orientation, and $(A',B\,',C')$ has positive orientation. While in the rotation with the center of rotation O which maps ΔPQR into the $\Delta P'Q'R'$, (P,Q,R) has a positive orientation and $(P',Q\,',R')$'s orientation is also positive.

Therefore

Definition

i. A transformation T maintains an orientation if for any three points which are not collinear (P,Q,R) have the same orientation as the orientation of the triple points (P',Q',R')

ii. A transformation T reverses an orientation if the orientation of any three points which are not collinear (P,Q,R) is not the same as the orientation of (P',Q',R') where $P' = T(P)$, $Q' = T(Q)$, and $R' = T(R)$.

Moreover, we can classify isometry into direct isometry and indirect isometry (opponent isometry) by looking at the following definition:

Definition
A transformation refers to direct isometry if the transformation preserves the orientation, and it is called opponent isometry if the transformation reverses the orientation.

An isometry refers to direct isometry if the isometry preserves orientation and called opponent isometry if the isometry changes the orientation. Thus, since the reflection changes the rotation, it means that the reflection is opponent isometry, while the rotation is a direct isometry since it maintains orientation.

Furthermore, exactly one property holds, i.e. any isometry is either direct isometry only or an opponent isometry only (not both).

Example 1 :

T is a transformation defined by $T\ (P)\ =\ (x-7, y\ +\ 4)$ for all points $P\ (x, y) \in V$. Show whether T is an isometry?

Solution:

Suppose that $P' = T(P)\ = (x-7, y+4)\ \forall\ P(x,y)\ \in\ V$

Suppose also that Point $A(x_1, y_1)$ and (x_2, y_2) with $A \neq B$.

It means that $A' = T(A) = (\ x_1 - 7, y_1 + 4)$ and

$B' = T(B) = (x_2 - 7, y_2 + 4)$

$AB = \sqrt{(x_2 - \mathrm{x}_1)^2 + (y_2 - y_1)^2}$

$A'B' = \sqrt{\{(x_2 - 7) - (\mathrm{x}_1 - 7)\}^2 + \{(y_2 + 4) - (y_1 + 4)\}^2}$

$= \sqrt{(x_2 - \mathrm{x}_1)^2 + (y_2 - y_1)^2}$

Since $A'B' = AB$, so T isometry

Example 2

Suppose T is a transformation that is defined for all points $P(x,y)$ as $T(P)\ =\ (-\ y, x)$.

a. Is T an isometry?

b. If T isometry, is it direct isometry or opponent isometry? Answer:

Solution:

a. Given $P' = T(P) = (y, -x)\ \forall\ P(x,y)\ \in V$

Suppose that point $A(x_1, y_1)$ and (x_2, y_2) with $A \neq B$.

$AB = \sqrt{(x_2 - \mathrm{x}_1)^2 + (y_2 - y_1)^2}$

$A'B' = \sqrt{(y_2 - y_1)^2 + \{(-x_2) - (-\mathrm{x}_1)\}^2}$

$= \sqrt{(y_2 - y_1)^2 + (-x_2 + \mathrm{x}_1)^2}$

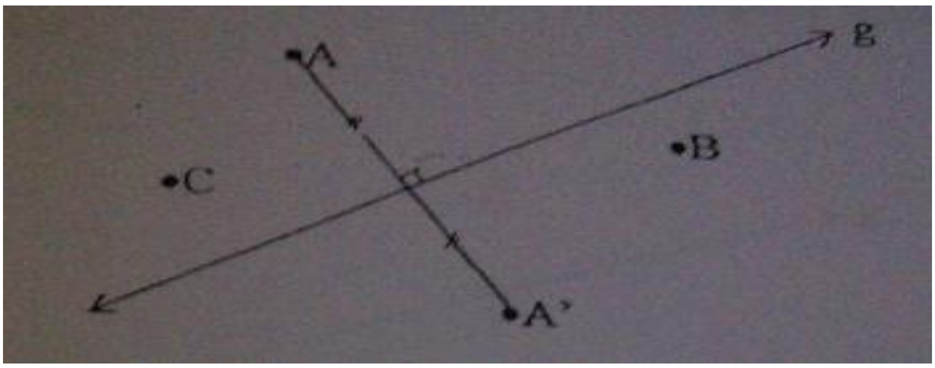
$= \sqrt{(x_2 - \mathrm{x}_1)^2 + (y_2 - y_1)^2}$

evidently $A'B' = AB$

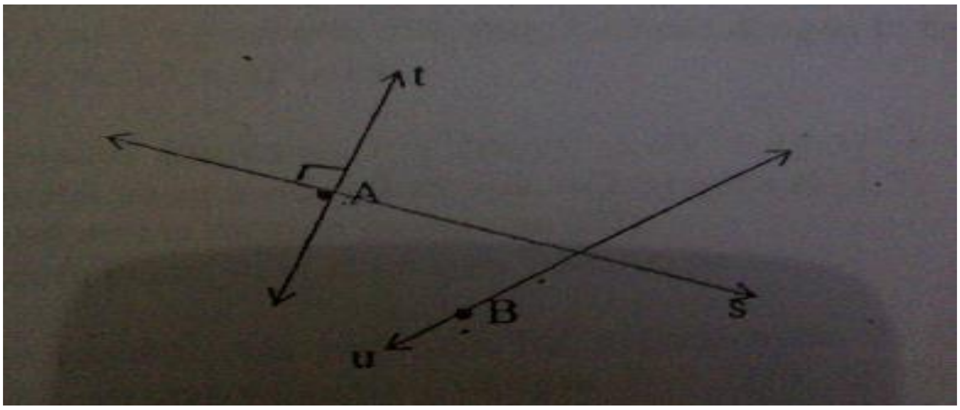
Because $A'B' = AB$, so T isometry.
b. To check whether T is a direct isometry or opponent isometry, take any three points which are not collinear, for example $O(0.0)$, $A(1,2)$ and $B(4.1)$.
By using transformation $T(P) = (y, -x)$) $\forall\, P(x, y) \in V$ then it is obtained that $O' = (0,0)$, $A' = (-2,1)$, and $B' = (-1,4)$.
Since the orientation (O, B, C) is positive, and the orientation (O', A', B') is also positive, then T is a direct isometry.

Exercises

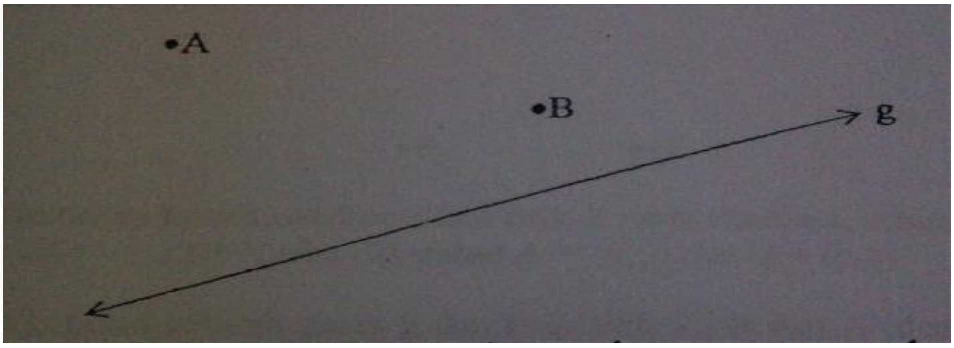
1. T is a transformation defined by $T(P) = (3x + 5, 2 - 4y)$ for all points $P(x, y) \in V$.
   Show that T is an isometry.
2. Suppose the points $A(1, -1)$, $B(4,0)$, $C(-4,1)$ and $D(-2, p)$. If an isometry T with $T(A) = C$ and $T(B) = D$, find the value of p.
3. Prove that the transformation with the formula:
   $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3/5 & -4/5 \\ -4/5 & -3/5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ is an isometry.
4. Suppose ΔABC by isometry T is mapped into $\Delta A'B'C'$, prove that $\Delta ABC \cong \Delta A'B'C'$.
5. Given a line g. T a function defined for each point of P in the plane V as follows:
   i. If $P \in g$ then $T(P) = P$
   ii. If $P \notin g$ then $T(P) = P'$ such that P' is the midpoint of the line segment of orthogonal from P to g.
      a. Is T a transformation?
      b. Is T an isometry?
      c. If there are two points A and B so that $A'B' = AB$ where $A' = T(A)$, $B' = T(B)$, what can be interpreted about A and B?
6. Suppose the S line and the points A, A', B and C with $A' = M_g(A)$. By only using a ruler without ascale, sketch point $B' = M_g(B)$ and $C' = M_g(C)$.



7. Given lines s, t, u; points A and B as in the following figure. T is an isometry where $B = T(A)$ and $u = T(s)$. If $t \perp s$, painting $t' = T(t)$



8. Suppose a line g and and a circle I. Prove that $M_g(I) = I'$ with I' is also a circle.
9. Suppose lines g, g', h, h' and k where $g' = M_k(g)$ and $h' = M_k(h)$. If $g' // h'$ prove that $g // h$.
10. Suppose lines g, h, and h' where $h' = M_g(h)$. Verifying the truth of the following expressions:
    a. If $h' // h$, then $h // g$
    b. If $h' = h$, then $h = g$.
    c. If $h' \cap h = \{A\}$, then $A \in g$.
11. Suppose line g and two points A and B as shown in the following figure:



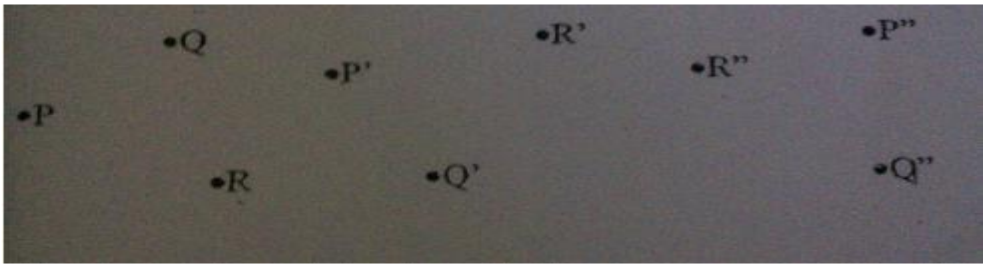
    a. by using an appropriate isometry, determine a point $P \in g$ so that $AP + PB$ as short as possible.
    b. If $Q \in g$ is different to the point P, prove that $AQ + QB > AP + PB$.
12. Suppose a circle $I = \{(x, y) \mid (x - 2)\ 2 + (y - 3)\ 2 = 4\}$. T is an isometry mapping point $A(2,3)$ on $A'(1,7)$.

    Find the equation of the set $T(I)$.

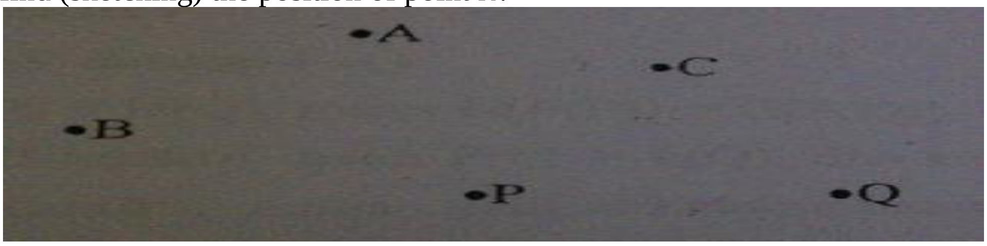
    Is the map of I also circle? Why?

13. In the following figure, there are three points which are not collinear, it is P, Q, R. T and S are isometry where $P' = T(P)$, $Q' = T(Q)$, $R' = T(R)$, and $P" = S(P)$, $Q' = S(Q)$, $R' = S(R)$.

    What kind of isometry T and S are those?



14. Isometry T maps point A to P, point B to Q and C to R. If T is an opponent isometry, find (sketching) the position of point R.



15. Find the coordinates of the point P on the x axis, measure of $\angle APO = \angle BPX$, if $A = (0,3)$ and $B = (6,5)$

16. Suppose line g and points A, B and A' where $A' = M_g(A)$, and $\overleftrightarrow{AB} // g$. By using only one ruler, determine the coordinate of point $B' = M_g(B)$.

CHAPTER IV

COMPOSITION OF TRANSFORMATIONS

A. Composition Two Transformations

If F and G are respectively a transformation, then the composition of the two transformations is defined as follows:

> Definition
>
> Suppose F and G are two transformations where F: V → V and G: V → V, then the composition of F and G are notated G.F defined as (G.F) (P) = G [F (P)], ∀ P ∈V.

What about the composition of F and G, is it a transformation? To answer the question, whether the composition of two transformations is also a transformation, then the following steps must be solved consecutively:

1. Checking whether the composition of two transformations is functions
2. Checking whether the composition of two transformations is surjective
3. Checking whether the composition of two transformations is injective

Therefore, suppose the composition of F and G is H, or H = G. F.

- If F and G are functions, it is clear that H is also a function.
- Is H surjective?

  Take arbitrary Y∈V. Is there X∈V such that H (X) = Y? Since G is transformation, for every Y∈V there is Z∈V such that G (Z) = Y. Similarly, since F is transformation then for every Z∈V, there is X∈V that F (X) = Z.

  From $G(Z) = Y$, it is obtained that $G[F(X)] = Y$ or $(G \cdot F)(X) = Y$, so $H(X) = Y$ meaning that H is surjective.
- Is H injective?

  Take arbitrary P, Q elements of V where $P \neq Q$

  Suppose $(P) = H(Q)$, then $G[F(P)] = G[F(Q)]$

  Since G is injective then $F(P) = F(Q)$, and since F injective then $P = Q$. It is contrary to which it is known that $P \neq Q$. It means that the assumption that $H(P) = H(Q)$ is not true. Therefore, it should be $H(P) \neq H(Q)$ implying H is injective.

  Since H is surjective as well as injective, then H is bijective. Thus, H is a transformation.

> Theorem
>
> The composition of two transformations is a transformation.

Example 1

Suppose that transformation $T1[(x,y)] = (x+2,y)$ and $T2[(x,2y)$. If T is the composition of T_1 and T_2, find T.

Solution:

T is the transformation T_1 and T_2 then :

$$T[(x,y)] = (T_1 \cdot T_2)(x,y)$$
$$= T_2[T_1(x,y)]$$
$$= T_2[(x+2,-y)]$$
$$= (x+2,-2y)$$

So the transformation T is $[(x,y)] = (x+2,-2y)$.

The Properties of The Composition of The Transformation

It has been discussed previously that the composition of two transformations is a transformation. It means that the composition of transformations is closed. Furthermore, the composition of transformations is also associative, but not commutative.

> Theorem
>
> If T_1, T_2, and T_3 are the transformation then $T_1[T_2.T_3]=[T_1.T_2].T_3$

B. Inverse Transformation

If $M_g(P) = P'$, then $M_g.M_g(P) = P$ or $M_g{}^2(P) = P$.

So, M^2 is a transformation that describes each point onto itself. This transformation is called the identity transformation is symbolized by the letter I.

So that, $I(P) = P, \forall P$.

If T is a transformation, then $T.I(P) = T[I(P)] = T(P), \forall P$.

So, $T.I = I$ and $I.T(P) = I[T(P)] = T(P), \forall P$.

Thus $I.T = T$. Implied $T.I = I.T = T$

So that,the identity transformation I is the number 1 in the set of transformations with multiplication operation among these transformations. In the set of real numbers, by multiplication operations, each transformation T has inverse Q such that T.Q = I = Q.T.

If there is an invers transformation of T, then the transformation of T is written T^{-1},so that $T.T^{-1}=T^{-1}.T = I$

Theorem Each transformation T has an inverse

<u>Proof:</u>

Suppose T is a transformation. We define the equivalent F as follows:

Suppose X is an element of V, where V is an euclidean field. Since T is a transformation, then T is bijective implying that A$\in V$, such that $T(A) = X$.

We define then $F(X) = A$. meaning that $F(X)$ is the pre-image of X, such that from $T(A) = X$, $T[F(X)] = X$ or $(T.F)(X) = I(X)$, for every X element of V. So $FT = I$, so that $TF = FT = I$.

Now it shall be proved that L is a transformation. From this definition, it is clear that F is surjective. Suppose $F(X_1) = F(X_2)$ and suppose $T(A_1) = X_1, T(A_2) = X_2$ where $F(X_1) = A_1$ and $F(X_2) = A_2$

Since T is a transformation, then $A_1 = A_2$, and it is also obtained that $X_1 = X_2$ such that F is injective. Thus, it proves that F is injective.

So F is a transformation.

The transformation F is called the inverse transformation of T and it is denoted by $F = T^{-1}$.

Theorem Transformation has exactly one inverse.

<u>Proof:</u>

Suppose T is a transformation with two inverses, namely S_1 and S_2 then

$(T \cdot S_1)(P) = (S_1 \cdot T)(P) = I(P)$ for all P and

$(T \cdot S_2)(P) = (S_2 \cdot T)(P) = I(P)$ for all P

so, $(T \cdot S_1)(P) = (T \cdot S_2)(P) \Rightarrow T\,[S_1(P)] = T\,[S_2(P)]$

because T is injective , then $S_1(P) = S_2(P)$ for all P

> Theorem
>
> Inverse of any reflection in a line is a reflection itself.

So, $S_1 = S_2 = S$

<u>Proof:</u>

A reflection in the line g is M_g

For $X \epsilon g$ then $Mg(X) = X$, So $M_g \cdot M_g(X) = M_g(X) = X = I(X)$

So, $M_g \cdot M_g = I$

Thus , the $Mg^{-1} = Mg$

For $X \notin g$, $Mg(X) = X'$, so the g axis $\overline{XX}$, then $M_g \cdot M_g(X) = M_g(X') = X = I(X)$

with g axis $\overline{XX}$. So, $M_g \cdot M_g = I$ atau $M_g^{-1} = M_g$

> Definition
>
> A transformation which has inverse which is the transformation itself is called an involution.

> Theorem
>
> If T and S are transformations, then $(T \circ S) = S^{-1} \circ T^{-1}$

<u>Proof:</u>

Since $(T.S)^{-1}$ is the inverse of (T.S) then $(T.S)^{-1}.(T.S) = I$. Meanwhile $(S^{-1}.T^{-1})(T.S) = S^{-1}.T^{-1}.T.S = S^{-1}.I.S = S^{-1}.S = I$

Since every transformation has only one inverse, then $(T_O S)^{-1} = S^{-1}\,T^{-1}$

Therefore the inverse of the composition transformations is the composition of the inverses of each transformations in reverse order.

Example 1

In an orthogonal axis XOY system, transformations F and G are defined as follows

For ∀ P (x, y), F (P) = (x + 2.1 / 2 y) and G (P) = (x-2,2y). (FG) (P) = F [G (P)] = F [(x-2,2y)] = (x, y) = P

While (GF) (P) = G [F (P)] = G [(x + 2.1 / 2 y) = (x, y) = P. So (FG) (P) = (GF) (P) = P = I (P), ∀ P or FG = GF = I

Thus F and G are the mutual transformations of each other denoted by $G = F^{-1}$ or $F = G^{-1}$

Example 2

On an orthogonal axis system, the line g = {f (x, y) | y = x} and h = {(x, y) y = 0}

Find P such that (Mh.Mg) (P) = R with R (2,7)

Solution:

Let P (x, y)

$(M_h.M_g)(P) = R \Rightarrow M_h (M_h.M_g)(P) = M_h(R)$

$\rightarrow (M_h M_h M_g)(P) = M_h(R)$

$\rightarrow (M_h M_h)(M_g)(P) = M_h(R)$

$\rightarrow (M_g)(P) = M_h(R)$

$\rightarrow M_g.M_g(P) = M_g.M_h(R)$

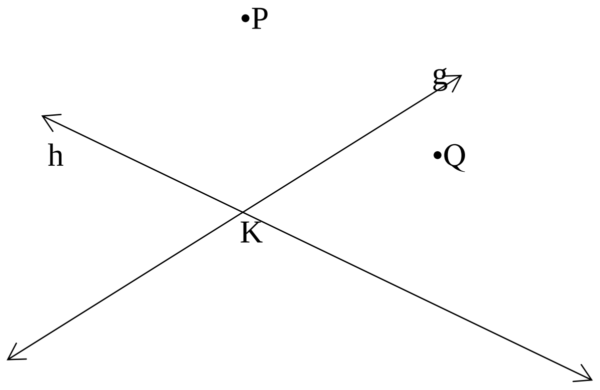
$\rightarrow P = M_g.M_h(R)$

implying that $P(x,y) = M_g.M_h(2,7) = M_g(2,-7) = (-7,2)$.

So the coordinate of the point P is (-7,2)

Exercises

1. Given lines g and h. A point K is the intersection of g and h, as well as the points P and Q points on g and h. Sketch:



a. $A = M_g [M_h(P)]$
b. $B = M_h [M_g (P)]$
c. $C = M_h [M_h (P)]$
d. $D = M_g [M_h (K)]$
e. R so $M_h [M_g (R)] = Q$
f. Do $M_g.M_h = M_h.M_g$? Why?

2. Suppose that T and S are isometries, check whether the following statements below are true and give the reason.
   a. TS is an isometry
   b. TS = ST
   c. If g is a line, then g' = (TS) (g) is also a line.
   d. If g//h, and g' = (TS) (g), h'=(TS) (h), then g'//h'

3. Given two intersecting lines g and h, skecth
a. K such that M_g [Mb (k)] = g
b. b m such that $M_h [M_g (m)] = g$
c. N such that $M_h [M_g (n)]$ line divides the acute angle between g and h.
4. The line g is the x-axis of an orthogonal axis system and h {(x, y | y = x}. Define:
a. The line equation $M_h [M_g (g)]$
b. P " = $M_h [M_g (P)]$, with P (0,3)
c. Q''= M_g [Mb (Q)], with Q (3,-1)
d. R''= $M_{g.}$ M_h (R) with R (x,y)
e. The magnitude of <RQR "when O is the origin
5. Let g is x-axis, and h = {(x, y | y = x}. S is a mapping defined as follows. If P∈g then S (P) = P, And if P∈g then S (P) is the midpoint of the perpendicular line from P to G.
a. Prove that S is a transformation.
b. If P (x, y) any point, determine the coordinate of point S.M_g (P)
c. Check whether S. $M_g = M_g$.S.
d. Check whether S. $M_h = M_h$.S.
6. If g = (x, y) | y = 0} and h {(x, y) | y = x} and S is a transformation defined as question 5, whereas A (2, -8) and P (x, y) , Determine the coordinates of the following points
a. M_g M_h S(A).
b. M_g.S .M_h (A).

c. $S\ M_g.S.\ (A)$.
d. $M_h.S\ .M_g\ (P)$.
e. $S^2.M_h\ (P)$.
f. $S\ M_g^2(P)$

7. Suppose that g and h are two lines which are perpendicular to each other. A, B, and C are three points so that Mg (A) = B and Mh (A) = C. Determine the following points.

a. $M_g^3(A)$
b. $M_h\ M_g\ M_h\ (A)$
c. $M_h\ M_g\ M_h\ M_h\ M_g\ (A)$
d. $M_g^2\ M_h^3\ (A)$

8. Simplify.!

a. $(W_gV_hM_g)^{-1}$
b. $(M_h\ V_h\ W_g\ M_g)^{-1}$

9. Suppose the transformations $T_1\ [(x,y)] = (-x,y)$ and $T_2\ [(x,y)]= (x,\frac{1}{2}y)$. Find the formula for $T_2.\ T_1$ then if $T\ _1= T_2.\ T_{1,}$ find the equation $T\ (g)$ if $g = \{(x,\ y) \mid x + y = 0\}$. What is $T_2.\ T_1 = T_1.\ T_2$?
10. If two different lines g and h intersect at point P, prove that $M_g\ M_h\ (A) = P$ if and only if A = P
11. It is known that g // h and points P, Q are neither on g nor on h.

a. Sketch $P'' = M_g\ M_h(P)$ and $Q''= M_g\ M_h(P)$
b. What is the form of quadrilateral PP’’Q’’Q ?
c. Prove your opinion!

CHAPTER V

HALFTURN

Halfturn is a special case of rotation, where the rotation angle is 180°. Since, halfturn has special characteristic, it is discussed earlier. In the previous section, it has been discussed that a reflection is an involution. Another example of an involution is a halfturn surrounding a point. One halfturn reflects every point in a plane figure at a certain point.

Therefore, halfturn is called a reflection about point, and that point is the center of the halfturn.

Definition

Halfturn about a point A is a mapping S_A that is defined for each point P on a plane as follows :

i. If $P = A$, then $S_A(P) = P$.

ii. If $P \neq A$, then $S_A(P) = P'$, where A as the center point of $\overline{PP'}$

Since halfturn is also a reflection of a point, and reflection is a transformation, then it can be said that halfturn is a transformation.

Theorem

Halfturn is a transformation.

Suppose $A(a,b)$,S_A map point $P(x,y)$ to $P'(x',y')$, then $S_A(P) = P'$ where A is the center point of $\overline{PP'}$ so $\frac{x+x'}{2} = a$ and $\frac{y+y'}{2} = b$.

It is obtained that $x' = -x + 2a$ and $y' = -y + 2b$.

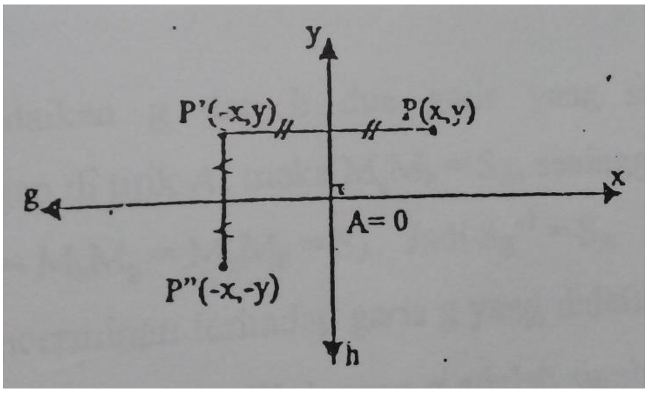
Thus, if $A = (a,b)$and $P\ (x,y)$, then $S_A(P) = (2a - x, 2b - y)$.

Theorem

If g and h intersect and are perpendicular at point A, then $S_A = M_g M_h$.

Proof :

Since $g \perp h$, we can make a system of orthogonal axis where g as axis x and h as axis y, and A is used as the point of origin. It must be proved that each point $P\ (x, y)$, satisfies $S_A(P) = M_g M_h(P)$.



Let $P\ (x, y) \neq A$and $S_A(P) = P''(x_1, y_1)$.

Since A is the center point $\overline{PP''}$, so $(0,0) = \left(\frac{x_1+x}{2}, \frac{y_1+y}{2}\right)$, so $x_1 = -x$ and $y_1 = -y$, therefore, $S_A(P) = (-x, -y)$.

While $(M_g M_h)(P) = M_g[M_h(P)] = M_g[(-x, y)] = (-x, -y)$

It turns out that $S_A(P) = (M_g M_h)(P) = (x, y)$

Thus, $S_A = M_g M_h$

Theorem

If g and h are two lines perpendicular to each other, then $M_g M_h = M_h M_g$

Proof

If $P = A$ (Look at the figure of the previous theorem), so $M_g M_h(P) = M_g(P) = P$. Whereas $M_h M_g(P) = M_h(P) = P$, so $M_g M_h(P) = M_h M_g(P)$. if $P \neq A$, then $M_g M_h = S_A$, while $M_h M_g(P) = M_h[(x, y)] = (-x, -y) = S_A(P)$

It turns out that $M_g M_h = M_h M_g = S_A$

So,$M_g M_h = M_h M_g$.

Theorem

If S_A is halfturn, then $S_A^{-1} = S_{A.}$

Proof:

Suppose that g and h are two lines perpendicular to each other and intersect at point A, then $M_g M_h = S_A$, implying that $S_A^{-1} = \left(M_g M_h\right)^{-1} = M_h^{-1} M_g^{-1} = M_h M_g = M_g M_h = S_A$. So $S_A^{-1} = S_{A.}$

Reflection in line g which is defined as $M_g(P) = P$ when $P \in g$, and $M_g(P) = P'$. with g is the axis $\overline{PP}$ when the $P \notin g$. When we look in general, so for every $P \in g$ the image of point P is the point itself. Such point is called invariant point of the reflection.

Definition:

Point A is called the invariant transformation of T, if it is satisfied that $T(A) = A$

It can be seen that reflection has many invariant points that are infinite, meanwhile halfturn just has one invariant point, i.e. the center of the halfturn.

It has been discussed in the previous section that isometry is a transformation which maps a line into a line. When a line by a transformation has image in the form of a line, such transformation is called a collineation.

Based on the above understanding, any isometry is a collineation. Since halfturn is an isometry, it is also a collineation. Among collineations, one of them is dilatation, defined as follows :

Definition:

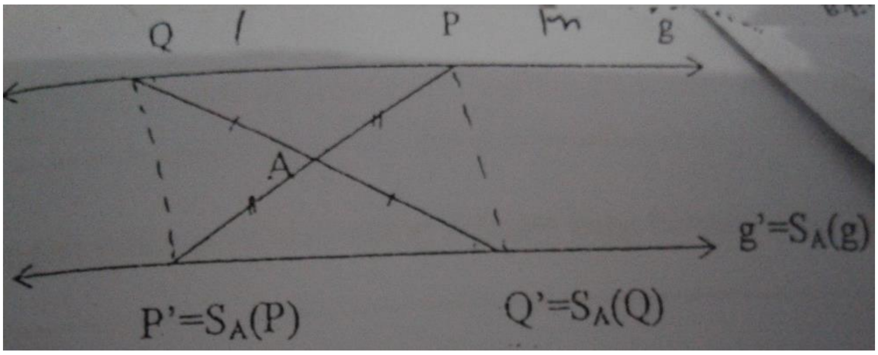
A collineation (Δ) is called dilatation, if for every line g, it satisfies the property $\Delta g // g$.

One example of collineation which is dilatation is halfturn. The example was verified by the following theorem :

Theorem:

Suppose S_A is a halfturn and g is a line, if $A \notin g$, then $S_A(g) // g$

Proof :



Suppose $P \in g$ then A is the midpoint of the segment $\overline{PP'}$ with $P' = S_A(P)$.

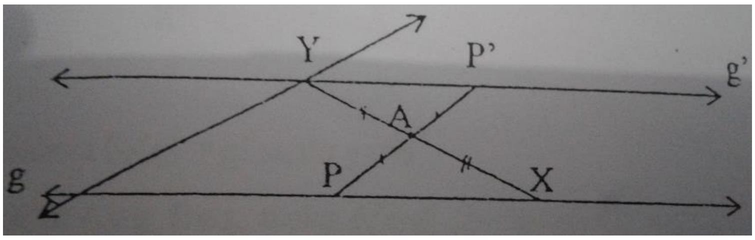
Suppose $Q \in g$ then $A$is the midpoint of the segment $\overline{QQ'}$ with $Q' = S_A(Q)$.

Since $\Delta APQ \cong \Delta AP'Q'$ then $PQP'Q'$ a parallelogram, implying that PQ // $P'Q$". Thus, g // $S_A(g)$.

Example 1 :

Suppose that two lines g and h are not parallel, A is a point not located neither on g nor h. Determine all points X on g and all points Y on h such that A is the midpoint of the segment $\overline{XY}$.

Solution :



Take a point $P \in g$. Sketching $P' = S_A(P)$. Then $g' = S_A(g)$ passes through P' where $PA = AP'$, g' // g. If g' intersecting h in Y, then draw a line YA intersecting g in X. Then X and Y are the pair of the points which seems to be exactly the only one pair

a. Prove that X and Y is the only pair that satisfy the condition.

b. If we didn't use $g' = S_A(g)$ but we used $h'' = S_A(h)$, could we get another pair?

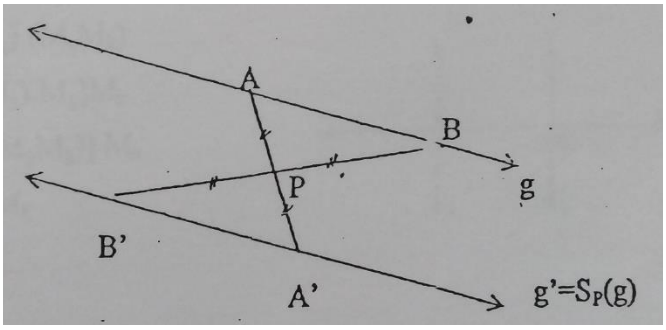
Theorem :

A halfturn is a dilatation which has involutoric characteristic.

Suppose P is the center of a halfturn, and g is a line. We must prove that:

a. $S_P(g)$ // g

b. $S_P S_P = I$, where I is an identity transformation



Prove:

a. It means that $S_P(g) = g'$ is a line.

Suppose $A \in g, B \in g$, then $A' \in g'$, and $PA = PA'$, $PB = PB'$, while $m\,(\angle APB) = m(\angle A'P'B')$. Since $\triangle\, PAB \cong \triangle\, PA'B'$, $ABA'B'$ is a parallelogram implying $g' // g$.

b. Since $S_PS_P(A) = S_P(A') = A$ for all points $A \in g$, then $S_PS_P(g) = I(g)$.

So $S_PS_P = I$ meaning that S_P is an involution.

The Composition of halfturns

The characteristics of the composition of halfturns are classified according to their centers and whether there is an invariant point.

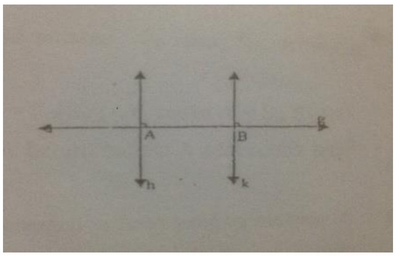
Theorem

The composition of two halfturns with different centers does not have invariant point.

Proof:

Suppose A and B are the centers of the halfturns. Suppose $g = \overline{AB}$, both h and k are perpendicular line to $\overline{AB}$ in A and B respectively, then:

$$\begin{aligned} S_AS_B &= (M_hM_g)(M_gM_k) \\ &= [(M_hM_g)M_g]M_k \\ &= [M_h\,(M_gM_g)]M_k \\ &= M_h\,I\,M_k \\ &= M_hM_k \end{aligned}$$



Suppose that X is the invariant point of S_AS_B, then $S_AS_B(X) = X$ implying $(M_hM_k)\,(X) = X$ or $M_k(X) = M_h(X)$.

Suppose $M_k(X) = X_1$

If $X \neq X_1$, then each h and k is $\overline{XX_1}$ axis and since a line segment has exactly one axis, it must be $h = k$. It is certainly not possible since $A \neq B$.

If $X = X_1$ then $M_k(X) = X_1$ and $M_h(X) = X_1$, So $X \in h$ and $X \in k$ meaning that k and h intersect at point X which is not possible because h // k.

Indeed, there can not be a point X such that $M_K(X) = M_h(X)$ or $S_A S_B = X$.

So $S_A S_B$ doesn't have a fixed point.

Theorem
If A and B are two different points, then there is only one and a halfturn that maps A to B.

Proof:

Suppose there are two halfturns S_D and $S_{E,}$ so $S_D(A) = B$ and $S_E(B) = A$.

So, $S_D(A) = S_E(B)$, then $S_D[S_D(A)] = S_D[S_E(A)]$, then $A = S_D S_E(A)$. If D and E are two different points, it means that A is a fixed point $S_D S_E . I$ which is not possible. Therefore, there is no more than one halfturns that maps A to B. The only halfturn is $S_T(A) = B$ where T is the midpoint of $\overline{AB}$.

Example:

Given $E = \{(x, y)|x^2 + 4y^2 = 16\}$, $A(4, -3)$, and $B(3,1)$.If g is the X axis, show that $A \in M_g S_B(E)$?

Solution:

It is known that $(M_g S_B)^{-1} = \mathrm{S}_B{}^{-1} . \mathrm{M}_g{}^{-1} = \mathrm{S}_B M_g$

If $P(x, y)$, then $M_g(P) = (x, -y)$ then $\mathrm{S}_B(P) = (2.3 - x, 2.1 - y) = (6 - x, 2 - y)$

$A \in M_g S_B(E) \Leftrightarrow S_B M_g(A) \in E$

$S_B M_g(A) = S_B\left[M_g(4, -3)\right] = S_B(4,3) = (2, -1)$

Since $(2, -1) \notin E$, then $\left(M_g S_B\right)^{-1}(A) \notin (E)$

In a similar way, we can define a set of maps if the equations are known.

In the latest example we know that $P \in M_g S_B(E)$ if and only if $\left(M_g S_B\right)^{-1}(P) \in (E)$.

If $P(x, y)$ so $\left(M_g S_B\right)^{-1}(P) = (6 - x, 2 + y)$, then $\left(M_g S_B\right)^{-1}(P) \in (E)$ if and only if $(6 - x, 2 + y) \in \{(x, y)|x^2 + 4y^2 = 16\}$.

So it must be $(6-x)^2+4(2+y)^2=16$. $P(x,y)\in M_gS_B(E)$ if and only if $P(x,y)\in \{(x,y)|x^2+4y^2-12x+16y+36=0\}$

Therefore $x^2+4y^2-12x+16y+36=0$ is the equation of the image of E by transformation of M_gS_B.

Exercises

1. Suppose three distinct points A, B, P are not collinear, sketch
   a. $S_A(P)$
   b. $S_AS_B(P)$
   c. $R\ so\ S_B(R)=P$
   d. ${S_A}^2(P)$
2. Given the line g and point A, $A\notin g$
   a. Draw lines $g'=S_A(g)$, why $S_A(g)$ is a line?
   b. Prove that $g'//\ g$
3. Given ΔABC and a parallelogram WXYZ. There is a point K which lies outside the triangle ΔABC and the parallelogram WXYZ.
   a. Sketh $S_{\mathrm{K}}(\Delta\mathrm{ABC})$!
   b. Find a point J so $\mathrm{S_J(WXYZ)=WXYZ}$
4. If A = (2,3) determine!
   a. S_A (C) if C (2,3)
   b. S_A (D) if D (-2,7)
   c. S_A^{-1} (E) if E (4, -1)
   d. S_A (P) if P (x, y)
5. If C = (-4.3) and g = {(x, y) | y = x}, determine!
   a. $M_g S_c$ (2.4)
   b. $M_g S_c$ (P) if P = (x, y)
   c. $(M_g S_{c)})^{-1}$ (P)
   d. Is $M_g S_c = S_c M_g$? Explain
6. Give the implication from the following expressions :
   a. S_A (k) = S_A (j)
   b. S_A (D) = S_B (D)
   c. S_A (E) = E
   d. g is a line and S_A (g) = g
   e. If A ≠ B and $S_A S_B$ // g
7. Given A = (0,0) and B = (-4.1). Determine K such that the $S_A S_B$ (K) = (6,2)
8. Given A = (-1,4), g = {(x, y) | y = 2x-1} and h = {(x, y) | y = -4x}
   a. Determine the set equation S_A (g) = g '.
   b. Determine the set equation S_A (h) = h '.
   c. Determine the set equation S_A (axis-x)

d. Does the point (-5.6) lie on $S_A(g)$? Explain!

9. Given circle C = $\{(x, y) \mid x^2 + (Y-3)^2 = 4\}$, a line g = $\{(x, y) \mid y = x\}$ and A = (3,2). Show whether the point D = (2,5) is the element of the set $M_g S_A(C)$.

10. Given line g, point P, and circle C, and suppose $P \neq g$ does not intersect the C and $P \notin C$. Moreover, C circle centered at A

a. By applying a halfturn, construct the line segment $\overline{AT}$, so that the $X \in C$, $Y \in g$ such that the P midpoint of $\overline{XY}$.

b. Prove that the construction is correct.

CHAPTER VI

TRANSLATION

A. Vector

We have previously learned line segment in other lectures. In this part, we will discuss about vector which is defined as follows

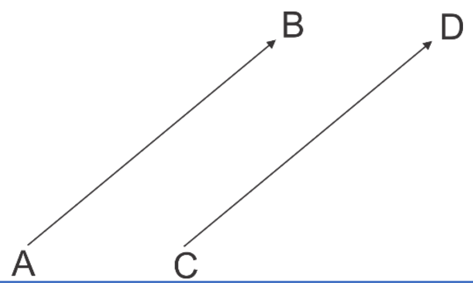
> **Definition**
>
> Vector is a line segment of which one of the edge is the initial point and the other edge is the terminal point.

The notation of vector

- The notation of vector AB is $\overrightarrow{AB}$
- The notation of vector CD is $\overrightarrow{CD}$

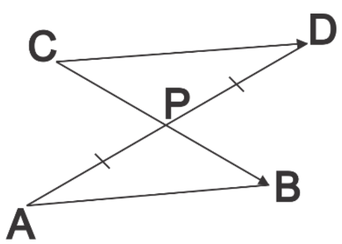
  VectorAB and CD are vector of which the points A and C are the the initial points, and the points B and D are the terminal points.

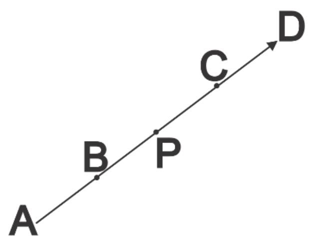
  As the illustration, look at the following figure:



> **Definition**
>
> $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$ that is notated as $\overrightarrow{AB} = \overrightarrow{CD}$, if there is a halfturn $S_p(A) = D$ where P is the midpoint $\overline{BC}$

The definition above can be interpreted also that $\overrightarrow{AB} = \overrightarrow{CD}$, if $S_p(C) = B$ and P is the midpoint of $\overline{AD}$, as an illustration, look at the following figure:





The equivalence of two vectors satisfies a characteristic asserted by the following theorem

Theorem

If there are two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ which are not collinear then $ABCD$ is a parallelogram if and only if $\overrightarrow{AB} = \overrightarrow{CD}$

Proof:

Look at the latest figure:

1. Suppose that $\overrightarrow{AB} = \overrightarrow{CD}$, if P is the midpoint of $\overline{BC}$, then $S_p(A) = D$, by the definition of equivalence. The diagonals of quadrilateral $ABCD$ are bisected in P. Then $ABCD$ is a parallelogram.
2. Suppose that $ABCD$ is a parallelogram, then diagonals AD and BC are intersected in the midpoint P implying that $S_p(A) = D$. In other words, P is the midpoint of both AD and BC.

So $\overrightarrow{AB} = \overrightarrow{CD}$

Consequence:

If $\overrightarrow{AB} = \overrightarrow{CD}$, then $ABCD$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are either parallel or colinear

Theorem

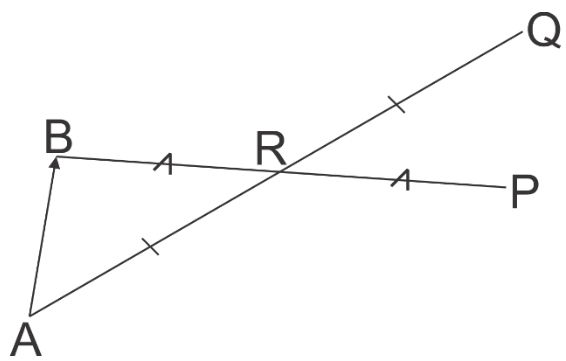
Suppose $\overrightarrow{AB}$, $\overrightarrow{CD}$ dan $\overrightarrow{EF}$ are vectors, then the following properties apply:s

- $\overrightarrow{AB} = \overrightarrow{AB}$ $(reflexive)$
- If $\overrightarrow{AB} = \overrightarrow{CD}$ then $\overrightarrow{CD} = \overrightarrow{AB}$ $(symmetric)$
- If $\overrightarrow{AB} = \overrightarrow{CD}$ then $\overrightarrow{CD} = \overrightarrow{EF}$ then $\overrightarrow{AB} = \overrightarrow{EF} (transitive)$ are colinear.

Theorem

If $\overrightarrow{AB}$ is a vector and there is a point P, then there is a unique point Q such that $\overrightarrow{PQ} = \overrightarrow{AB}$.

Proof :



Suppose R is the midpoint of $\overrightarrow{BP}, Q = S_R(A)$ then $\overrightarrow{AB} = \overrightarrow{PQ}\ or\ \overrightarrow{PQ} = \overrightarrow{AB}.$

To prove the uniqueness of point, let $\overrightarrow{AB} = \overrightarrow{PT}.$

Then $S_R(A) = T$, because R is thee midpoint of $\overrightarrow{BP}$ and the image of A by S_R, then $T = Q$ meaning that $\overrightarrow{PQ}$ is the only vector where P is the initial point and Q is the terminal point which is equivalent to $\overrightarrow{AB}$

Corollary 1:

If $P_1(x_1, y_1), P_2(x_2, y_2)$, and $P_3(x_3, y_3)$ are points with the specified coordinates, then $P(x_3 + x_2 - x_1\ , y_3 + y_2 - y_1)$ is the only point that satisfies $\overrightarrow{P_3P} = \overrightarrow{P_1P_2}$

Corollary 2:

If $P_n = (x_n, y_n), n = 1,2,3,4$ then $\overrightarrow{P_1P_2} = \overrightarrow{P_3P_4}$. If and only if $x_2 - x_1 = x_4 - x_3$, and $y_2 - y_1 = y_4 - y_3$.

The last concept that is related to the vector is a scalar multiplication by vector. Suppose that $\overrightarrow{AB}$ is vector and k is real number, so:

a. If $k > 0$, then $k\overrightarrow{AB}$ is a vector $\overrightarrow{AP}$ which is defined as $\overrightarrow{AP} = k(\overrightarrow{AB})$, $P \in \overrightarrow{AB}$ (ray AB).
b. If $k < 0$, then $k\overrightarrow{AB}$ where P is the opposite ray of $\overrightarrow{AB}$ and $\overrightarrow{AP} = |k|(\overrightarrow{AB})$

B. Translation

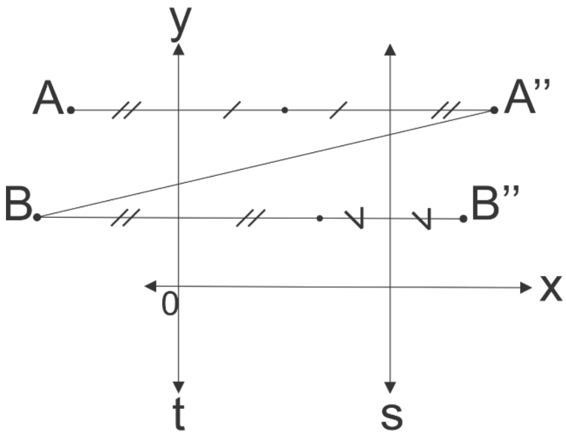
In this section, the concept of translation is introduced using the definition of vector.

Theorem

If s and t are two parallel lines, then A and B are the two points, so $\overrightarrow{AA''} = \overrightarrow{BB''}$ with $A'' = M_t M_s(A)$ and $B'' = M_t M_s(B)$.

Proof.

Choose a coordinate system with t as the y-axis and a line which is perpendicular to t as the x-axis, and s which its parallel to t.



Suppose $A = (a_1, a_2)$ and $B = (b_1, b_2)$.

If $N\ is$ the midpoint of $\overrightarrow{A''B}$, then we have to prove $S_N(A) = B''$.

If the equation of s is $x = k(k \neq 0)$, then

$A'' = M_s M_t(A) = M_s(-a_1, a_2) = (2k + a_1, a_2)$

$B'' = M_s M_t(B) = M_s(-b_1, b_2) = (2k + b_1, b_2)$

Because we know that N is the midpoint of $\overrightarrow{A''B}$, then

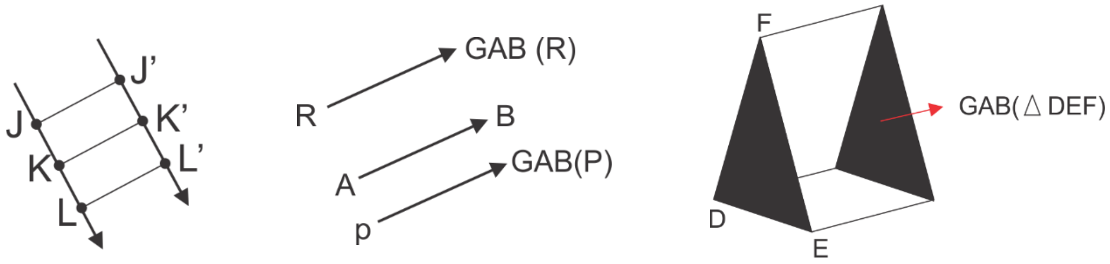
$N = \left[\frac{(2k + a_1) + b_1}{2}, \frac{a_2 + b_2}{2}\right]$, whereas

$S_N = \left[2\left[\frac{2k + a_1 + b_1}{2}\right] - a_1, 2\left[\frac{a_2 + b_2}{2}\right] - a_2\right] = (2k + b_1, b_2)$

Evidently $S_N(A) = B''$. Therefore $\overrightarrow{AA''} = \overrightarrow{BB''}$

Definition

A mapping G is a translation, if there is a vector $\overrightarrow{AB}$ such that for every point P in a plane V has image $P'$where $G(P) = P'$and $\overrightarrow{PP'} = \overrightarrow{AB}$.



In the figure above, it is clear that every vector determines a translation.

If$\overrightarrow{AB}$ is a vector then G_{AB} is a symbol to address a translation in the length of AB.

Theorem

If $AB = CD$ then $G_{AB} = G_{CD}$

Proof:

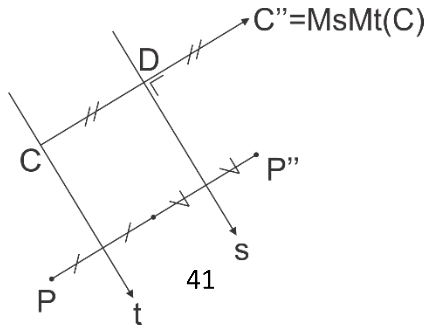
If P is an arbitrary point, it must be proved that $G_{AB}(P) = G_{CD}(P)$. Suppose $G_{AB}(P) = P_1$ and $G_{CD}(P) = P_2$, so $PP_1 = \overrightarrow{AB}$ and $PP_2 = \overrightarrow{CD}$. Since $\overrightarrow{AB} = \overrightarrow{CD}$, $\overrightarrow{PP_1} = \overrightarrow{PP_2}$ meaning that $P1 = P2$ and $G_{AB} = G_{CD}$

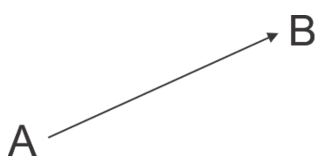
Theorem

Suppose t and s as the two lines is parallel and $\overrightarrow{CD}$ is the vector that perpendicular to s and t, with $C \in s$ and $D \in s$. If $\overrightarrow{AB} = 2\overrightarrow{CD}$ so $G_{AB} = M_s M_t$

Proof:

Suppose P is an arbitrary point, if $P' = G_{AB}(P)$ and $P'' = M_s M_t(P)$, so it is necessary to prove that $P' = P''$



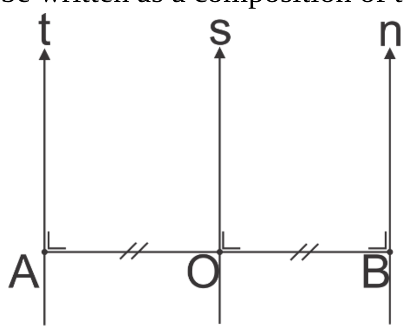


As the property of translation suggests, if $G_{AB}(P) = P'$ then $\overrightarrow{PP'} = \overrightarrow{AB}$. Since, $\overrightarrow{AB} = 2\overrightarrow{CD}$, $\overrightarrow{PP'} = 2\overrightarrow{CD}$.

Related to $C'' = M_s M_t(C), C \in t$ so $C'' = M_t(C)$. It means that D is the midpoint of $\overline{CC''}$ implying that $\overrightarrow{CC''} = 2\,\overrightarrow{CD}$. Since $\overrightarrow{CC''} = \overrightarrow{PP''}$, $\overrightarrow{PP''} = 2\,\overrightarrow{CD} = \overrightarrow{PP'}$, and it means that $P' = P''$, so $G_{AB} = M_s M_t$.

Notes

1) Each translation G_{AB} can be written as the composition between two reflections in two lines which are perpendicular to $\overline{AB}$ and it is ½ AB in length.
2) If AB is a line and C is the midpoint of $\overline{AB}$ whereas t, s and n are the perpendicular lines to AB in A, C and B respectively, then $G_{ab} = M_s M_t = M_n M_s$
3) Since any translation can be written as a composition of two reflections, whereas



a reflection is a transformation is isometry then a translation is an isometry transformation.

Translation is a direct isometry

Theorem

If G_{AB} is a translation, then $(G_{AB})^{-1} = G_{BA}$

Proof:

According to latest figure, $G_{AB} = M_s M_t = M_n M_s$

While $G_{BA} = M_t M_s = M_s M_n$

$(G_{AB})^{-1} = (M_s M_t)^{-1} = M_t^{-1}.Ms^{-1} = M_t M_s = G_{BA}$

So, $(G_{AB})^{-1} = G_{BA}$.

C. The Closeness of translation

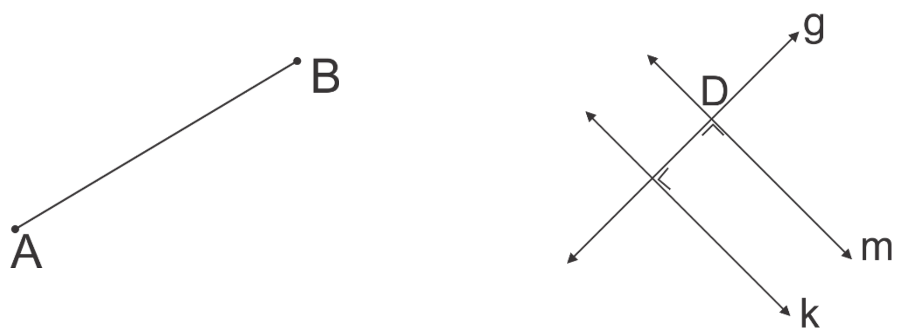
In the previous section, it is explained that a translation can be expressed in the form of a composition of two reflections. This section will describe that the composition of the two translations is a translation as well. The assertion refer to the following theorem:

Theorem

If G_{AB} is a translation, C and D are points such that $\overrightarrow{AB} = \overrightarrow{CD}$, then $G_{AB} = S_p S_C$.

Proof :

Suppose that $g = \overleftrightarrow{CD}, k \perp g \ in \ C, m \perp g \ in \ D$



AB is a vector from k to m, therefore $\overrightarrow{AB} = 2\overrightarrow{CD}$, so $G_{AB} = S_D S_C$. While $S_D = M_m M_g$ and $S_c = M_g M_k$.

Then $S_D S_C = (M_m M_g)(M_g M_k) = M_m\big(M_g M_g\big)M_k = M_m M_k . So \ G_{AB} = S_D S_C$.

Example 1

Given $A = (3,-1), B = (1,7)$ and $C = (4,2)$. Find the coordinate of a point D such that $G_{AB} = S_D S_C$.

Solution:

Suppose that E is a point such that $CE = AB$, then $E[4 + (1-3), 2 + (7-(-1))]$ or

$E = (2,10)$. If D is the midpoint $\overline{CE}$, then $D = (3,6)$, implying that $\overrightarrow{CE} = 2\overrightarrow{CD}$.

Thus, $\overrightarrow{AB} = 2\overrightarrow{CD}$, which obtains $G_{AB} = S_D S_C$ where D $(3,6)$.

Theorem

The composition of a translation and a halfturn is a halfturn.

Proof:

Suppose that G_{AB} is a translation and C is any point and suppose that E is also a point so

$\overrightarrow{CE} = \overrightarrow{AB}$. If D mid point $\overline{CE}$ then $\overrightarrow{CE} = 2\overrightarrow{CD}$.

According to previous theorem $G_{AB} = S_D S_C$, then $G_{AB} S_C = S_D S_C S_C = S_D I = S_D$. So $G_{AB} S_C = S_D$.

As a result of the theorem above is:

If S_A, S_B and S_C is half round, then $S_A S_B S_C = S_D$, with D is points that satisfy $\overrightarrow{AD} = \overrightarrow{BC}$.

To declare the composition of two translations is a translation in Cartesian coordinates, consider the following theorem.

Theorem

If G_{OA} is a translation where the coordinates of point 0 and point A are respectively $(0,0)$ and (a, b). T is a transformation that maps each point $P\ (x, y)$ to $T\ (P) = (x + a, y + b)$ then $G_{OA} = T$.

Proof:

For $P(x, y), T(P) = (x + a, y + b)$. Suppose that $P' = G_{OA}(P)$ then $PP' = OA$ implying that $P' = (x + a - 0, y + b - 0) = (x + a, y + b)$. Thus $T(P) = G_{OA}(P)$ for each $P \in V$. In other words $G_{OA} = T$.

Example 2

Suppose that G_{AB} is a translation which takes point $A(2,3)$ to point $B(4,1)$ and G_{CD} is a translation which takes point $C(-3,4)$ to point $B(0,3)$. If $P(x, y)$. Determine the coodinate of $G_{CD} G_{AB}(P)$.

Solution:

Suppose that 0' $= G_{AB}(0)$ and 0'' $= G_{CD}(O)$ then $\overrightarrow{OO'} = \overrightarrow{AB}$ and $\overrightarrow{OO''} = \overrightarrow{CD}$.

Then 0' $= (0 + 4 - 2, 0 + 1 - 3) = (2, -2)$ and

0'' $= (0 + 0 + 3, 0 + 3 - 4) = (3, -1)$.

So $G_{AB}(P) = (x + 2, y - 2)$ and $G_{CD}(P) = (x + 3, y - 1)$.

Thus $G_{CD}G_{AB}(P) = G_{CD}[(x \pm 2, y - 2)]$

$$= (x + 2 + 3, y - 2 - 1)$$

$$= (x + 5, y - 3)$$

Exercises

1. Suppose there are 4 points that are wirtten as A, B, C and D of which each pair of three points is not collinear. Skecth the following:
   a. Point E such that $\overrightarrow{CE} = \overrightarrow{AB}$
   b. Point F such that $\overrightarrow{DF} = \overrightarrow{BA}$
   c. $S_A(AB)$

2. Given points $A(0,0), B(5,3)$, and $C(-2,4)$. Find the coordinate :
   a. R so that such that $\overrightarrow{AR} = \overrightarrow{BC}$
   b. S so that such that $\overrightarrow{CS} = \overrightarrow{AB}$
   c. T so that such that $\overrightarrow{TB} = \overrightarrow{AC}$

3. If $A(1,3), B(2,7)$, dan $C(-1,4)$ are the vertices of parallelogram $ABCD$. Determine the coordinate of the point D.

4. Suppose there are two lines g and h whcih are parallel, point $P \in g$, and point Q neither on g nor on h.
   a. Sketch $P' = M_hM_g(P)$ and $Q' = M_gM_h(Q)$
   b. Prove that $\overrightarrow{PP'} = \overrightarrow{QQ'}$

5. Given a line g and circles L_1 and L_2. Line g are not cuting the circles L_1 and L_2. Use a transformation to draw a square having vertices located on g, a vertex that is located on $L_1$and the other vertex points located on L_2

6. If $P_0 = (0,0), P_1 = (x_1, y_1), P_2 = (x_2, y_2)$, and $P_3 = (x_3, y_3)$ while $k > 0$.
   a. Find the coordinate of P such that $\overrightarrow{P_0P_1} = k\overrightarrow{P_0P_1}$
   b. Find the coordinate of P such that $\overrightarrow{P_1P} = k\overrightarrow{P_1P_2}$
   c. Is the statement "If $\overrightarrow{P_3P} = k\overrightarrow{P_1P_2}$, then $P = \{x_3 + k(x_2 - x_1), y_3 + k(y_2 - y_1)\}$." valid for $k < 0$?

7. Given A, B, and C which are not collinear. Sketch:
   a. $G_{AB}(A)$ and $G_{AB}(B)$.
   b. $G_{AB}(C)$.
   c. The lines g and h where $A \in g$ and $G_{AB} = M_hM_g$.

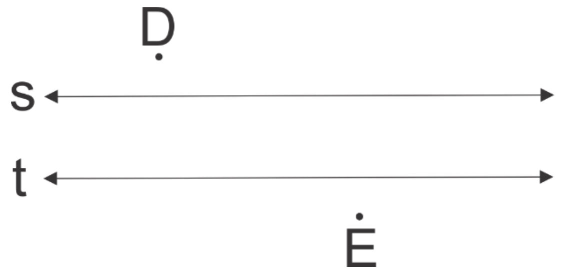
8. Suppose there are two points that notated as A and B and line g such that $g \perp \overrightarrow{AB}$. Sketch:
   a. Line h such that $M_hM_g = G_{AB}$.
   b. Line k such that $M_gM_k = G_{AB}$.
   c. Line m such that $m' = G_{AB}(m)$.
   d. Point C such that the $G_{BA}(C) = B$

9. Suppose lines g and h are parallel and there is point A which is not on the line.
   a. Draw the point B such that $M_hM_g = G_{AB}$
   b. Draw the point C such that $M_gM_h = G_{2AC}$

10. Given $A\ (2,3)$ and $B\ (-4,7)$, determine the equation of a line g and h such that $M_hM_g = G_{AB}$

11. Given three points $A\ (-1,3), B\ (5,-1)$ and $C\ (2,4)$
    a. Find the coordinate $C' = G_{AB}\ (C)$
    b. Find the equation of lines g and h so $C \in g$ and so $M_hM_g = G_{AB}$

12. The edges of a river are depicted with two parallel lines that is written as t and s (see figure). Above the river, there will be built a bridge and according to a good construction, the bridge must be made perpendicular to the direction of the river. Where is the bridge should be constructed usch that the distance fromthe town D to the town E will be as short as possible.

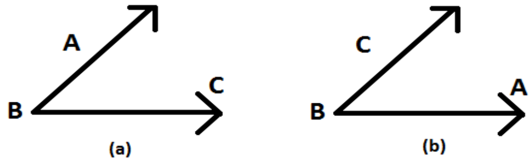


CHAPTER VII

ROTATION

A. Directed Angle

Angle has been introduced previously as the combination between two rays that have identical initial point. For example, angle ABC notated by ∠ ABC is formed by BA and BG rays. We can see angle ABC in the following figure:



Both figures (a and b) certainly illustrate angle ABC. But for the next discussion both figures will be distinguished. It is distinguished by using initial ray and terminal ray of an angle.

It's used to determine what type of positive angle or negative angle from an angle. Such angle is called directed angle

> **Definition**
>
> Directed angle is an angle of which one of the ray is the initial ray and the other is the terminal ray

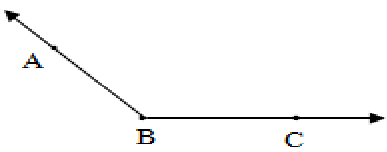
To symbolize an angle, for example ∠ ABC is a directed angle where $\overrightarrow{BA}$ rays is the initial ray and $\overrightarrow{BC}$ is the terminal ray notated as ∠ ABC

For another purpose, <ABC cannot be written as <BCA. For directed angle <CBA, the initial ray is $\overrightarrow{BC}$and the end ray is $\overrightarrow{BA}$.

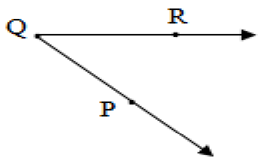
In Euclidean geometry, it has been studied about the magnitude of an angle, i.e. every angle ABC has magnitude in the interval from 0° to 180°, which is notated by $0^{\circ} \leq m < ABC \leq 180^{\circ}$.

It is the same as a directed angle $< ABC$ which always has a magnitude, however, it has following properties:

a) If the triple orientation (BCA) is positive, then $m < ABC \ = \ m \ < ABC$
b) If the triple orientation (BCA) is negative, then $m < ABC \ = \ -(m < ABC)$



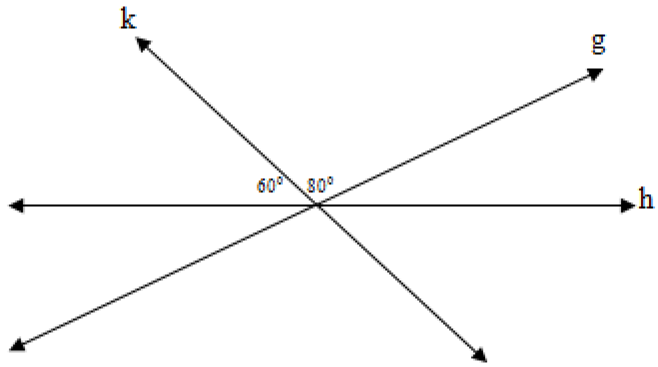
$m < ABC < 0$



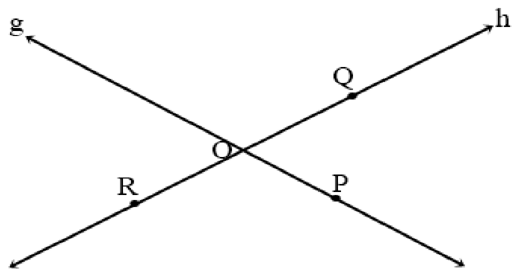
$m < PQR > 0$

When $< ABC$ is an angle then $< ABC \ = < CBA$, so $m < ABC \ = \ -m < CBA$. But for a directed angle $< \ ABC$ it applies for the $m < ABC =$
$-m \ CBA$ since the orientation of the BAC is always contrary to the orientation of the BCA.

If there are two intersecting lines not perpendicular to each other, then its angle is the acute angle.



In the figure, the magnitude of the angle between line g and k is 80 °, and the magnitude of the angle between h and k is -60 °.
The angle between two lines can be described as follows:
Suppose that g and h intersect at point O, point P is on line g, points Q and R are located on line h as suggested in the following figure:



If ∠POR is an acute angle, then the magnitude of line g to line h is equal to ∠POR, whereas if the angle ∠POR is an obtuse angle, then the angle from line g to line h equals the magnitude of ∠POQ. Suppose m∠POR = 140 °, then the magnitude from line g to line h is m∠POQ = 40 °, while the magnitude of line h to line g is m∠QOP = -40 °.

B. Rotation

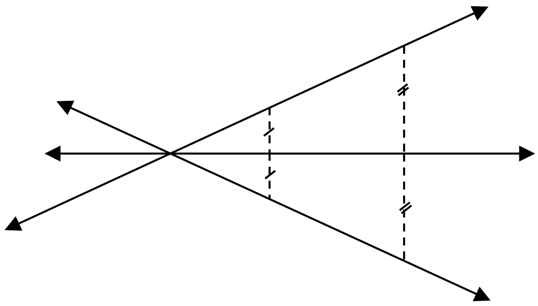
It will be discussed in the present section, the composition of the two reflection in the lines that are not parallel and in intersecting lines but are not perpendicular to each other. The composition of the two reflections will produce an isometry either in the form of a rotation and a translation. The composition is a basic theorem of rotation.

Theorem B

If s and t intersect at point A but are not perpendicular, the points P and Q are the points that are different from A, then m (< PAP") = m (< QAQ") with P" $=M_tM_s(P)$ and Q" $=M_tM_s(Q)$

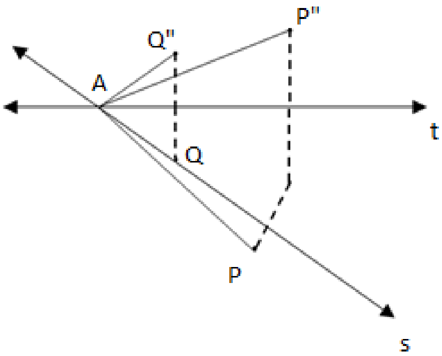
Proof:

Case 1 : P∈ s and Q∈ s



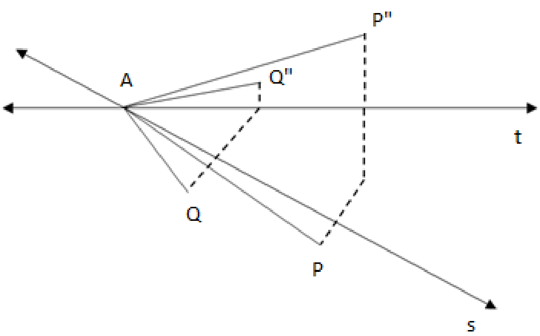
A"= $M_tM_s(A)$ means that A"=A. P"=$M_tM_s(P)$ and Q"=$M_tM_s(Q)$. Sincethe points A, P and Q are located on are collinear, A", P", and Q" are collinear. Thus, lines PP" and QQ" intersect at point A. Thus m (< PAP") = m (< QAQ").

Case 2 : P∉ s and Q∈ s



m (< PAP") = m (< PAQ) + m (< QAP"), then m (< QAQ") = m (< QAP") + m (< P"AQ"), since m (< PAQ)= m (< P"AP") so m (< PAP") = m (< QAQ").

Case 3 : P∉ s and Q∉ s



In the case 3, we can prove that if P”= M_tM_s(P) and Q”=M_tM_s(Q), so m (< QAQ” = m (< PAP”).

Therefore, by transformation of M_tM_s, every point rotates by the same directed angle, rotating the same point. Such process is called a rotation.

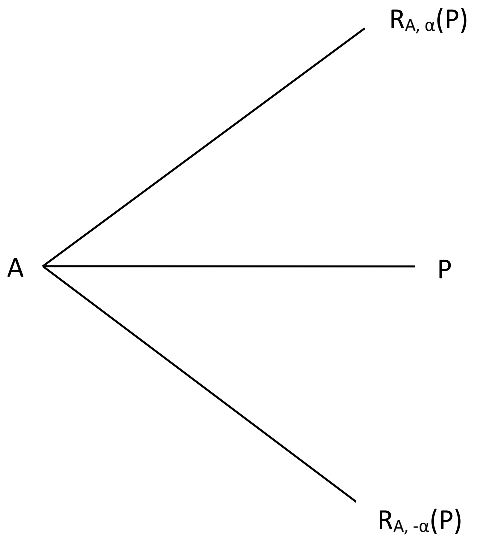
Defenition

If A is a point and α is angle where $-180°<\alpha<180°$, a rotation with center A and angle α denoted by $R_{A,\alpha}$ is a function from V to V defined by

1. If P = A, so $R_{A,\alpha}$(P) = P.
2. If P $\neq$ A, so $R_{A,\alpha}$(P) = P’, then m (< PAP”) = $\alpha$and AP’=AP.

Based on the definition, the rotation $R_{A,\alpha}$, just has an invarioant point, i.e. A (the center of the rotation). The image of point the P by rotation $R_{A,\alpha}$, is a point of a circle where A is the center and AP is the radius.

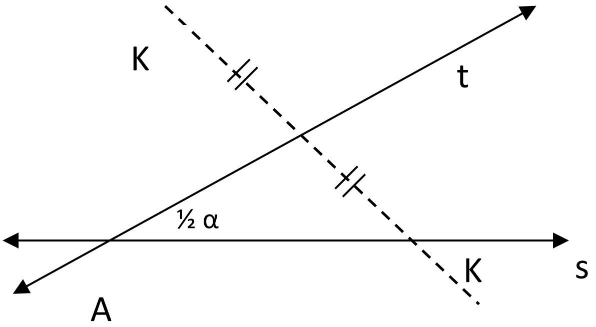
Since magnitude α of the angle rotation is between -180^0 and 180^0, so $\alpha>0$, if the direction of the angle is opposite to the clockwise direction, and $\alpha<0$ if the direction of the angle is the same as the of the clockwise direction.



Theorem

If s and t are two lines that intersect on A, s and t aren’t perpendicular, and if the magnitude of the angle from line s to t is $^{\alpha}/_{2}$, then $R_{A,\alpha}$ =M_tM_s

Proof:



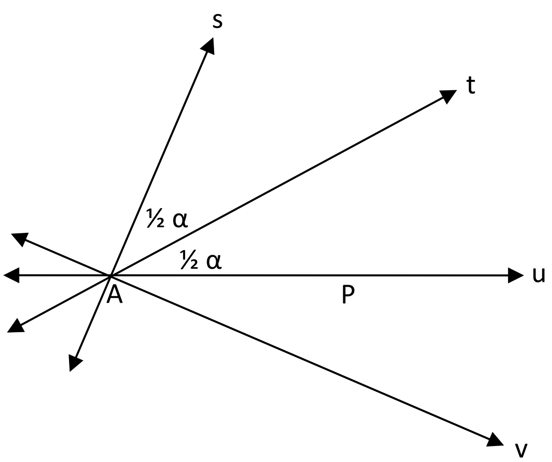
Suppose that a point K ≠ A located on s. If K' = M_tM_s(K) so m(<KAK') = 2. ½ α = α. Since <KAK'=α, so $R_{A,\alpha} = M_tM_s$.

> **Theorem**
> 1. Rotation with A as the center and the magnitude α ($R_{A,\alpha}$) is a transformation.
> 2. Every rotation is direct isometry
> 3. The composition of two reflections is either rotation or translation

Example

If $R_{A,\alpha}$ is a rotation that maps point P to P', determine two pairs of lines which can be used as some reflection axes such that the composition of these reflections is a rotation.

Solution



1. Suppose that s = $\overline{AP}$, t is the bisecting line of <PAP', and suppose the magnitude of angle from s to t is ½ α, so $R_{A,\alpha} = M_tM_s$.

2. Suppose u = $\overline{AP'}$, and v is a line passing through A, then the angle magnitude from u to v is ½ α, so $R_{A,\alpha} = M_v M_{u.}$

C. Rotation Composition

We have proved that the rotation is a transformation. Because the composition of two transformations is transformation, what about the composition of two rotations? Whether it is a rotation or another transformation? For the composition of two transformations, the following matters are discussed:

a. If the centers of the rotations are the same
b. If the centers of rotation are different

If the centers of the rotations are the same, the composition of two rotations is a rotation with the same center and the magnitude of the angle is the sum of the angles (provided that if the sum is greater than 180° it must be substracted by 360°while if the sum is less than -180° it must be added by 360°).

Meanwhile, the composition of the rotations having different centers, can be in the form of a rotation of which the center is different to the centers of th composed rotations . In addition, the angle magnitude is the sum of the angles of the rotations following the previous case order. If the sum of the angles of is zero, then the composition of two rotations forms a translation.

In general it can be conluded as follows:

If R_{A,α_1} andR_{B,α_2} is two rotation, then $R_{B,\alpha_2}R_{A,\alpha_1} = R_{c,\alpha}$ with the conditions of α as follows:

1. If $0° < |\alpha 1 + \alpha 2| < 180°\ then\ \alpha = \alpha 1 + \alpha 2.$
2. If $\alpha 1 + \alpha 2 > 180°\ then\ \alpha = (\alpha 1 + \alpha 2) - 360°.$
3. If $\alpha 1 + \alpha 2 < -180°\ then\ \alpha = (\alpha 1 + \alpha 2) + 360°.$

Example

$R_{A,120}\ .\ R_{A,30} = R_{A,150}$

$R_{B,160}\ .\ R_{B,40} = R_{B,-160}$.

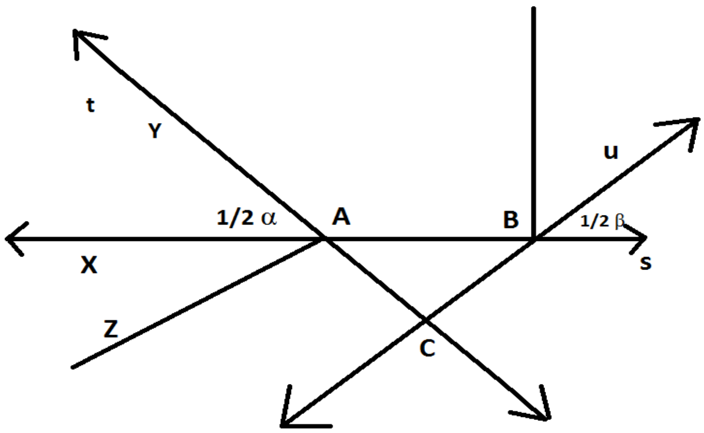
$R_{C,-150}\ .R_{C,-50}\ . = R_{C,160}$.

While if$\alpha 1 + \alpha 2 = 0°, then$ $R_{B,\alpha_2}R_{A,\alpha_1}$is a rotation .

Theorem

The composition of two rotations is a rotation or translation.

Proof :



Assume there are rotations $R_{A,\alpha}$ and $R_{B,\beta}$. Draw a line s = $\overleftrightarrow{AB}$, if m(∠XAY) = m(∠XAZ) = $\frac{1}{2}\alpha$, then $R_{A,\alpha}$ = M_sM_t and $R_{B,\beta}$ = M_uM_s.

So $R_{B,\beta}R_{A,\alpha}$ = $M_uM_sM_sM_t$= M_uM_t.

If u is parallel to t then $R_{B,\beta}R_{A,\alpha}$ is a translation, and if u and t intersect at C, then M_uM_t is a rotation centered at C. Assume $R_{C,0}$ = $R_{B,\beta}R_{A,\alpha}$then in α, β, dan θ, there is following relationship:

m(∠ABC) = $\frac{1}{2}\alpha$, m(∠BAC) = $\frac{1}{2}\beta$.

Therefore m(∠PCB) = $\frac{1}{2}\alpha + \frac{1}{2}\beta$. It means that the angle from t to u is $\frac{1}{2}\alpha + \frac{1}{2}\beta$, so that $\theta = \alpha + \beta$.

If $\alpha + \beta > 180°$, then θ= $(\alpha + \beta) - 360°$.

EXERCISES

1. Given points A and P are different. Construct the following
   A. $R_{A,90}$ (P)
   B. $R_{A,150}$ (P)
   C. $R_{A,45}$ (P)
   D. Q such that $R_{A,30}$ (Q) = P
2. Given m (<ABC) = 40˚ and m (<BAD) = 120˚, determine:
   A. m (<DAB),m(BCA), m(<ECA)
   B. The magnitude of angles fom $\underset{AB}{\leftrightarrow}$ to $\underset{BC}{\leftrightarrow}$, from $\underset{AC}{\leftrightarrow}$ to $\underset{BC}{\leftrightarrow}$, and from $\underset{AB}{\leftrightarrow}$ to $\underset{AC}{\leftrightarrow}$
3. Suppose point A and P are different, find m(<PAP') if P' is the image P by the following transformations:
   a. $R_{A,30}$, $R_{A,90}$
   b. $R_{A,-60}$, $R_{A,120}$
   c. $R_{A,135}$, $R_{A,90}$
   d. $R_{A,-120}$ $R_{A,-150}$
4. Simplify the following transformation compositions:
   A. $R_{A,30}$ $R_{A,60}$

B. $R_{A,120}$ $R_{A,-90}$
C. $R_{A,135}$ $R_{A,-90}$
D. $R_{A,-60}$ $R_{A,45}$
E. $R_{A,-120}$ $R_{A,-150}$
F. R_A,-120 R_A,90

5. Suppose there are two lines s and t which intersect at point A and two points P and Q not on the lines.
   A. Construct point $P' = M_t M_s(P)$
   B. Construct point $Q' = M_s M_t(Q)$
   C. Construct point $P'' = M_t M_s(P)$
   D. if m (<PAP") = 118° what is the angle magnitude between s and t.
6. Given points A, B and B 'such that RA, α (B) = B'. Construct two lines s and t such that MsMt = RA, α
7. Given 0 is the center point of the orthogonal coordinate system and A=(1,0). Find the coordinates of the following points:
   a. $R_{0,90}(A)$
   b. $R_{0,45}(A)$
   c. $R_{0,120}(A)$
   d. $R_{0,-135}(A)$
8. Writen the equation of lines s and t such that $M_s M_t$ equals the following rotation, if A=(1,3) and 0 is the center point.
   a. $R_{O,90}$
   b. $R_{O,-180}$
   c. $R_{O,120}$
   d. $R_{A,90}$
   e. $R_{A,-90}$
9. If A is the center point of an orthogonal coordinate system and $s = \{(x, y)|y = 2x - 3\}$ find the equation of $s' = R_{A,90}(S)$
10. If I is a circle with radius 2 and the center in A=$(\sqrt{2}, \sqrt{2})$ and if B = (0,0), find the equation of I'=$R_{A,90}(I)$

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